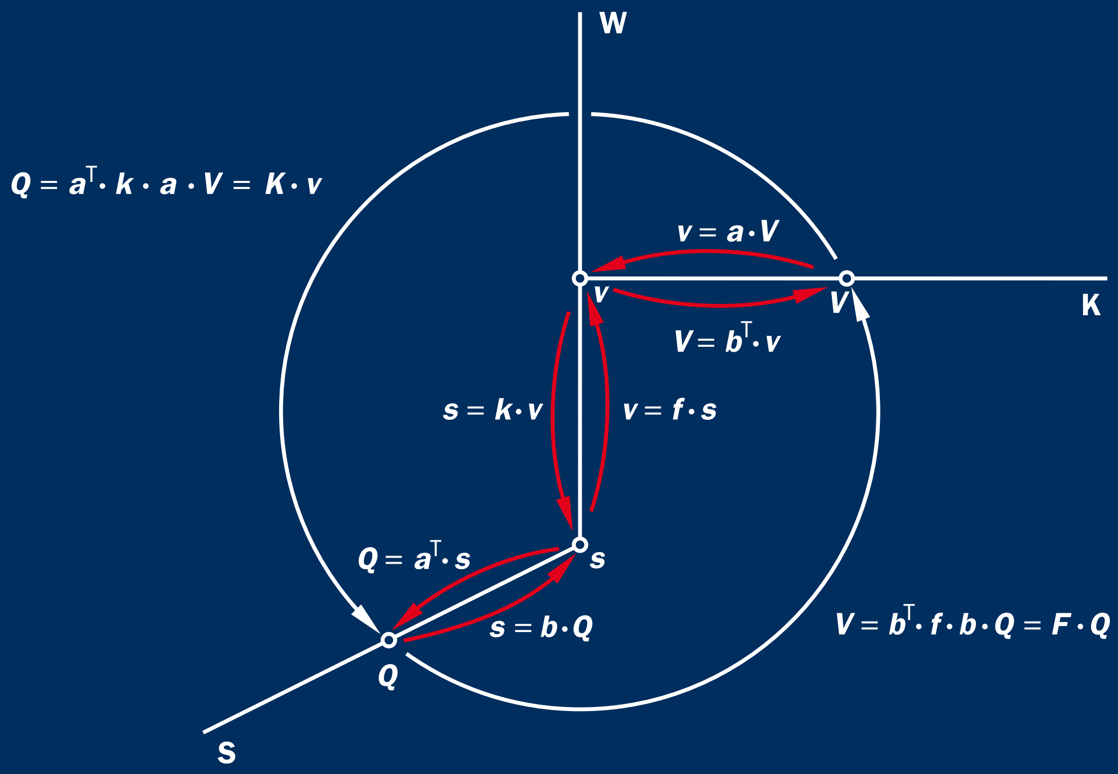
PETER MARTI

THEORY OF STRUCTURES

FUNDAMENTALS
FRAMED STRUCTURES
PLATES AND SHELLS



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EXAMPLECOLLECTION

5 STATIC RELATIONSHIPS

5.1 Force systems and equilibrium

5.1.1 Terminology

Forces are perceived through their effects. They correspond to physical interactions that cause or modify states of deformation or motion in material systems. The effect of a force depends on its point of application, its magnitude and its direction. Therefore, according to Fig. 5.1(a), a force can be represented as a point-based vector $\boldsymbol{F}$ with point of application A, magnitude F and line of action f.

The line of action f and an arbitrary reference point O define one plane. If we imagine a body connected to this one plane, then it is clear that $\boldsymbol{F}$ would cause a rotation of the body about the axis n perpendicular to the plane and passing through O. The tendency to rotate is proportional to the magnitude F and the distance a of force $\boldsymbol{F}$ from O. The *position vector* $\boldsymbol{r}$ of the point of application A of $\boldsymbol{F}$ expresses the tendency to rotate with the *moment*

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} \tag{5.1}$$

correctly in terms of magnitude and direction; $|\boldsymbol{M}| = Fa$ applies and the vectors $\boldsymbol{M}$, $\boldsymbol{r}$ and $\boldsymbol{F}$ constitute a right-hand screw, see Fig. 5.1(b). As can be seen, the moment $\boldsymbol{M}$ remains unaltered if force $\boldsymbol{F}$ is translated along its line of action f.

Every force $\boldsymbol{F}$ has a corresponding *reaction* $-\boldsymbol{F}$ with the same line of action. According to this so-called *reaction principle*, a force without its reaction cannot exist.

Remote forces (e. g. gravity) exhibit points of application different to those of their reactions; the interaction between two bodies with mass generally takes place without contact. But in the case of *contact forces* (e. g. support forces), the points of application of forces and reactions are geometrically identical (although not materially identical); the interaction between support and supported body comes about through contact – if the contact is eliminated, so the contact force disappears as well.

The *inertial forces* that must be considered in dynamics (see section 8.3.4) do not have any reactions. They do not correspond to any physical interactions, instead are mathematical auxiliary variables.

Contact forces are generally in the form of *surface forces* (*surface loads*) distributed over a finite area. The contact force related to the unit of surface area, the *force per unit area*

$$\boldsymbol{t} = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}A} \tag{5.2}$$

is also known as a *stress vector*, see Fig. 5.2(a) and section 5.2.1.

Similarly, remote forces distributed over a finite three-dimensional space are called *body forces* (*body loads*) with a *force per unit volume* of

$$\boldsymbol{q} = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}V} \tag{5.3}$$

see Fig. 5.2(b).

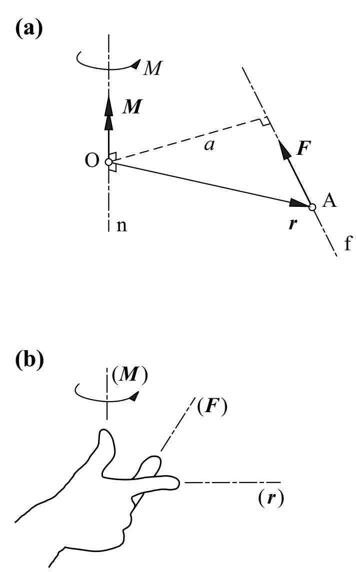

Fig. 5.1 Force and moment: (a) reference point O, point of application A, line of action f, axis of rotation n, (b) right-hand screw rule

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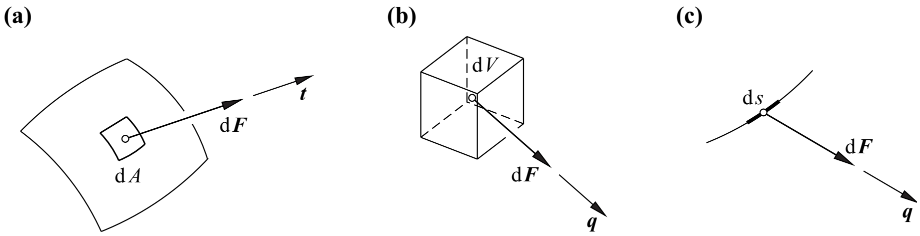

Fig. 5.2 Distributed forces: (a) force per unit area, (b) force per unit volume, (c) force per unit length

If, ultimately, a body such as a beam or cable is idealised as one-dimensional and if distributed forces act on this in the form of *line forces* (*line loads*), we get a *force per unit length*

$$\boldsymbol{q} = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}s} \tag{5.4}$$

see Fig. 5.2(c).

In the SI or MKS systems, the unit used for specifying the magnitude of a force is the *newton* [$1\,\mathrm{N} = 1\,\mathrm{mkgs}^{-2}$] or [kN] or [MN]. Correspondingly, the unit used for moments is [Nm] or [kNm] or [MNm]. To distinguish between forces and moments, we indicate the latter with double arrows, see Fig. 5.1(a). The units for forces per unit length, area and volume are therefore [Nm^{-1}], [Nm^{-2} = Pa] and [Nm^{-3}].

5.1.2 Force systems

We shall now consider *force systems* (groups of forces) whose material points of application lie within the arbitrary limits of a body or system. A body isolated from a body or system (or part thereof) by means of an imaginary cut is known as a *free body* (FB). By introducing all the forces that act on the free body, we obtain a *free-body diagram* (FBD).

The *resultant force* of a force system is obtained by adding together the vectors acting on the free body:

$$\boldsymbol{R} = \sum_{\mathrm{FB}} \boldsymbol{F} \tag{5.5}$$

Likewise, the *resultant couple* of a force system with respect to an arbitrary reference point O is

$$\boldsymbol{M}_{\mathrm{O}} = \sum_{\mathrm{FB}} \boldsymbol{r} \times \boldsymbol{F} \tag{5.6}$$

see Fig. 5.3 and (5.1).

If instead of O we select a different reference point O′, then according to Fig. 5.3, with $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{r}''$ and considering (5.5) and (5.6), it follows that

$$\boldsymbol{M}_{\mathrm{O}'} = \sum_{\mathrm{FB}} \boldsymbol{r}' \times \boldsymbol{F} = \sum_{\mathrm{FB}} \boldsymbol{r} \times \boldsymbol{F} - \boldsymbol{r}'' \times \sum_{\mathrm{FB}} \boldsymbol{F} = \boldsymbol{M}_{\mathrm{O}} - \boldsymbol{r}'' \times \boldsymbol{R} \tag{5.7}$$

The pair of vectors $\{\boldsymbol{R}, \boldsymbol{M}_{\mathrm{O}}\}$ or $\{\boldsymbol{R}, \boldsymbol{M}_{\mathrm{O}'}\}$ is called the *force-couple system* of the force system at O or O′.

Two force systems are *equivalent* when their force-couple systems are identical with respect to an arbitrary reference point. According to (5.7), the equivalence of two force systems need only be verified for one reference point; the identity of the resultant couple is then given for all points.

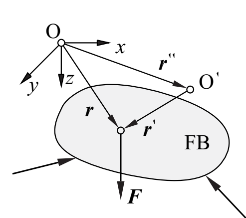

Fig. 5.3 Free body diagram with reference points O and O′

5.1.3 Equilibrium

A force system is *in equilibrium* when its force-couple system sums to zero:

$$\boldsymbol{R} = \boldsymbol{0} \quad , \quad \boldsymbol{M}_O = \boldsymbol{0} \tag{5.8}$$

The *equilibrium conditions* (5.8) result in six scalar equations in the case of force systems in three dimensions, i. e. three *force-balance equations* and three *moment-balance equations*. In the case of coplanar force systems, this number is reduced to three, i. e. two force-balance equations in the plane of the force system and one moment-balance equation perpendicular to that plane.

If (5.8) applies, then according to (5.7), $\boldsymbol{M}_{O'} = \boldsymbol{0}$. Consequently, the force-balance equations can be replaced by moment-balance equations about a second reference point. Generally, in the three-dimensional case, moment-balance equations can be formulated about six non-collinear axes and in the coplanar case about three points not lying in a straight line. In practice, this is often easier than setting up the force-balance equations. Depending on the particular problem, in the coplanar case only one, and in the three-dimensional case only one or two, force-balance equations are replaced by moment-balance equations, as is explained further in chapter 10.

Applying (5.8) to differential structural elements results in differential equations for the equilibrium, as dealt with in section 5.3.

When defining free bodies and applying the equilibrium conditions to those bodies, we generally use the so-called *free-body principle*: if we remove arbitrary parts from a compatibly deformed body or system in equilibrium by way of imaginary cuts, each one of those parts is in equilibrium and compatibly deformed.

Forces acting on arbitrary free bodies are known as *internal* or *external forces* depending on whether the material point of application of the reaction to a force lies inside or outside the free body.

According to the reaction principle, the internal forces form an *equilibrium system* (i. e. a force system in equilibrium), and so the external forces must themselves be in equilibrium if the free body is in equilibrium in its entirety. This assertion is known as the *fundamental theorem of statics*.

If the equilibrium conditions – at best following a suitable breakdown of the system – are sufficient for determining the unknowns in a problem, we speak of a *statically determinate* system, otherwise a *statically indeterminate* system.

5.1.4 Overall stability

Structures must be *stable*, i. e. they must not fail in their entirety (e. g. due to buoyancy, sliding or overturning). Their *rigid body equilibrium*, or rather their overall stability, must be assured (see section 4.4, limit state type 1).

Example 5.1 Cantilever retaining wall

The cantilever retaining wall shown in Fig. 5.4(a) is to be investigated for overturning about its toe O. To do this, we consider the cantilever retaining wall as a free body isolated from its surroundings according to Fig. 5.4(b) and add all the forces acting on it in order to create a free body diagram. Those forces are the dead loads of the base (G_1) (related to the unit length perpendicular to the yz plane) and the vertical stem (G_2), the surcharges due to the earth above the cantilevering parts of the base (G_3 and G_4), the active and passive earth pressures (E_a and E_p) plus a contact force A acting on the underside of the base. For simplicity, hydrostatic pressures are neglected. Further, the calculation with the earth surcharges G_3 and G_4 represents a considerable idealisation. Actually, in the event of an overturning failure, a wedge-shaped mass of soil would form in the ground behind the wall. And this would be linked with the mobilisation of further forces that are neglected here; a similar consideration applies to the soil in front of the wall.

The contact force A can easily be determined with (5.8) according to magnitude, direction and point of application, e. g. by setting up the two force-balance equations in the y and z directions and the moment-balance equation about O. Alternatively, A can also be determined graphically. Fig. 5.4(c) shows

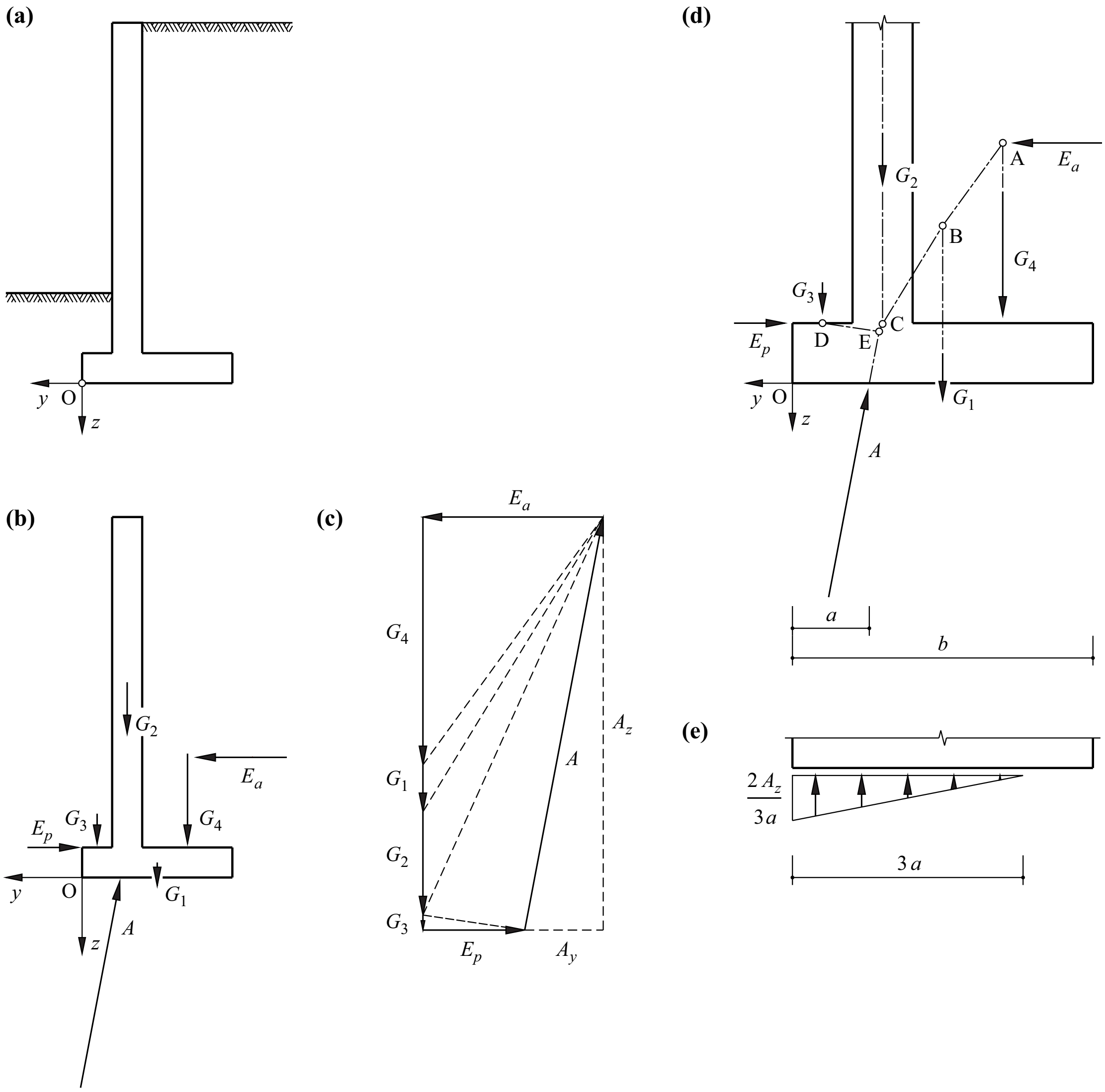

Fig. 5.4 Overall stability of a cantilever retaining wall: (a) overview, (b) free body diagram, (c) force polygon, (d) funicular polygon, (e) possible distribution of ground pressure

the associated addition of the force vectors in the so-called *force polygon*; for equilibrium, the force polygon must be closed, which determines the magnitude and direction of A. The point of application of A follows from the *funicular polygon* according to Fig. 5.4(d). This is done by successively forming the interim resultants (shown as dotted lines in the force polygon) of forces E_a and G_4 etc., drawing their lines of action starting from point A, the intersection of E_a and G_4, and extending this to intersect with the next force G_1 at B etc. This approach enables us to establish point E, the intersection of the lines of action CE and DE for (E_a, G_4, G_1, G_2) or (E_p, G_3), and hence determine the line of action of A.

For overall stability, A must act on the base, i. e. $0 \leq a \leq b$, see Fig. 5.4(d). For the limit case $a = 0$ (or $a = b$), the bearing pressure would be infinitely large, which is of course impossible because the strength of the subsoil is finite. Fig. 5.4(d) shows one practical possibility and Fig. 5.4(e) shows a statically equivalent linear bearing pressure distribution with a maximum value of $2A_z/(3a)$ at O. As can be seen, $3a < b$, i. e. in the range $-3a > y \geq -b$ the foundation experiences *partial uplift* with the contact force tending towards zero.

It is not possible to reach any conclusion about the distribution of the horizontal component A_y of A at the underside of the base solely on the basis of static considerations. For simplicity, a distribution proportional to A_z is assumed, which in this particular case means a triangular distribution.

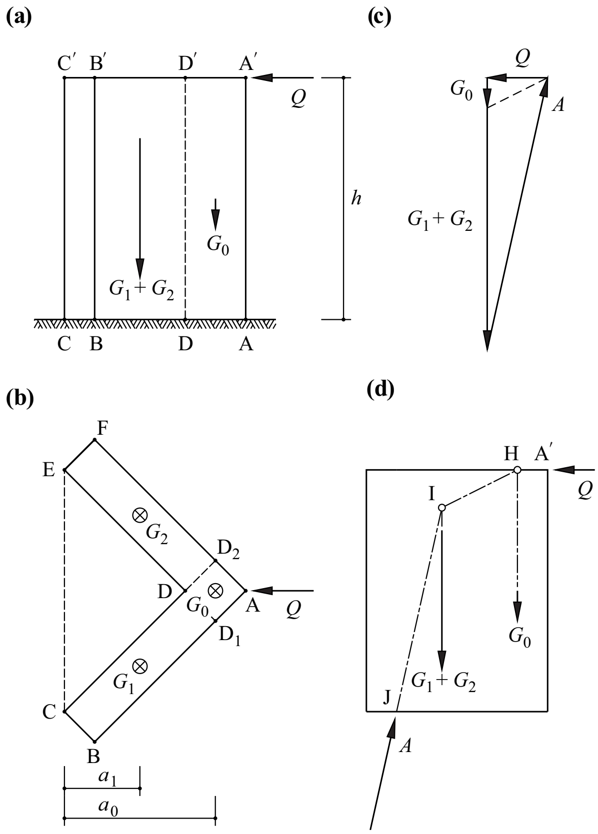

Fig. 5.5 Upturned right-angled element on horizontal surface: (a) elevation, (b) plan, (c) force polygon, (d) funicular polygon

Example 5.2 Support envelope

The upturned right-angled element supported on a horizontal surface shown in Fig. 5.5(a) and (b) is loaded at A′ by a horizontal force Q. In Fig. 5.5(b) we must distinguish between the *area of contact* ABCDEF and the *support envelope* ABCEF. The latter is the smallest convex envelope enclosing the former.

The overall stability of the right-angled element can be checked with the help of a moment-balance equation about axis CE. The overturning moment Qh due to Q acting about CE may not exceed the resisting moment $G_0 a_0 + (G_1 + G_2)a_1$ due to the dead load components $G_0(AD_1DD_2)$, $G_1(D_1BCD)$ and $G_2(DEFD_2)$, otherwise the element will overturn.

Fig. 5.5(c) and (d) show the alternative graphical examination with the help of the force and funicular polygons. For overall stability, the point of application J of the contact force A must lie within the support envelope.

The contact force A is assumed to be distributed equally over the end zones of the two legs of the element. In the limit case, the force is concentrated at points C and E, which means that the local bearing pressure is then infinitely large.

5.1.5 Supports

Supports correspond to the locally inhibited displacement and rotation capabilities (*degrees of freedom*) of structures. They can be classified according to the inhibited (restrained) displacement and rotation capabilities or the passive degrees of freedom, i. e. according to whether the displacements u, v, w and the rotations φ_x, φ_y, φ_z in the x, y, z directions are possible or prevented, see Fig. 5.6. The number of passive (restrained) degrees of freedom (or the number of components in the *support force-couple system*) is known as the *determinacy* of the support.

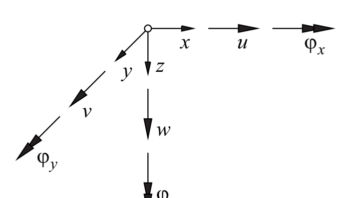

Fig. 5.6 Displacements and rotations

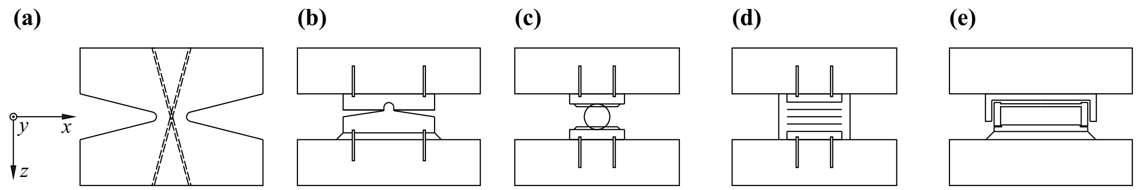

Fig. 5.7 Types of bearing: (a) concrete hinge, (b) steel linear rocker bearing, (c) steel roller bearing, (d) laminated elastomeric bearing, (e) elastomer pot sliding bearing

Fig. 5.7 shows a number of different types of support. If the concrete hinge shown in Fig. 5.7(a) is in the form of a linear support (long in the y direction), it inhibits displacements u, v, w and rotations φ_x, φ_z; in the form of a discrete support (short in the y direction), both φ_y and also φ_x and φ_z are practically not inhibited at all. It is notable that with respect to the forces, the support acts bilaterally in all three directions, i. e. positive and negative forces can occur in the body of the bearing sliced through at the horizontal joint, especially in the z direction, too, because of the reinforcement intersecting the horizontal joint. The steel linear rocker bearing shown in Fig. 5.7(b) acts bilaterally with respect to u and unilaterally with respect to w – the support would lift up in the z direction when the force tends towards zero; with respect to v, it works bilaterally up to a certain amount, either via friction or with lugs at the sides (after overcoming the play between lug and body of bearing); rotation φ_y is practically unrestrained, and rotations φ_x, φ_z are inhibited. In the case of the steel roller bearing shown in Fig. 5.7(c), u and φ_y are not inhibited and the support acts unilaterally with respect to w; guide rails at the side inhibit displacement v and rotation φ_z; rotation φ_x is inhibited because of the long roller in the y direction. The laminated elastomeric bearing shown in Fig. 5.7(d) functions unilaterally with respect to w and, depending on the particular type, enables displacements u, v as well as rotations φ_y, φ_x. The same is true for the elastomer pot sliding bearing shown in Fig. 5.7(e).

A closer look at Fig. 5.7 shows that, depending on the particular design, the displacement and rotation capabilities of supports always lie within certain limits and are never enabled or prevented in absolute terms. Likewise, the components of the support force-couple system associated with the inhibited displacement and rotation capabilities are restricted to certain limit values. In practice, it is certainly necessary to consider these limits carefully every time.

In theory of structures, we assume the appropriate idealisations shown in Fig. 5.8 for the coplanar case. Fig. 5.8(a) shows a unilaterally or bilaterally functioning *sliding support* (hinged support capable of displacement) that only inhibits w and whose support force-couple system is limited to the force component in the z direction. In the case of the *hinged support* shown in Fig. 5.8(b), u is also inhibited and the corresponding force component in the x direction is added to the support force-couple system. Considering the *fixed support* shown in Fig. 5.8(c), φ_y is finally inhibited as well; the support force-couple system also exhibits a moment about the y axis. Extending these considerations to the general three-dimensional case is easily possible with the help of Fig. 5.6.

Static equivalents to the types of support shown in Fig. 5.8 can be realised according to Fig. 5.9 with *pin-jointed members*. These are straight, weightless bars connected concentrically on both sides with frictionless hinges. With such assumptions, only forces can be transferred from the bars, whose lines of action coincide with the axes of the bars. So a statically equivalent substitute for a sliding support, as shown in Fig. 5.9(a), could be a pin-ended strut; the force component in the x direction caused by the inclination of the pin-ended strut as a result of a displacement u is negligible in comparison to the force component in the z direction, assuming infinitesimally small dis-

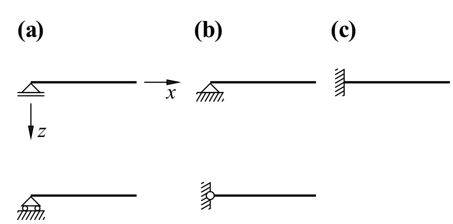

Fig. 5.8 Support idealisations: (a) sliding support, (b) hinged support, (c) fixed support

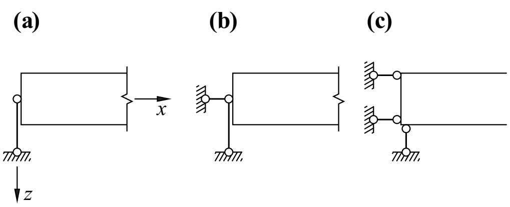

Fig. 5.9 Equivalent supports with pin-jointed members: (a) sliding support, (b) hinged support, (c) fixed support.

placements (first-order theory, see section 6.1). The support force component possible in the x direction with a hinged support requires a corresponding second pin-jointed member, as shown in Fig. 5.9(b). Ultimately, a third pin-jointed member is required to achieve fixity, as shown in Fig. 5.9(c); here, the first two pin-jointed members exhibit different lines of action and the axis of the third may not pass through the intersection of the first two, otherwise the support would not provide fixity, instead function like a hinged support at this point.

For the general three-dimensional case, six pin-jointed members are necessary for a braced support. With the force-couple system $\{\boldsymbol{R}, \boldsymbol{M}_\mathrm{O}\}$, the coordinates r_{ij} of one point on the axis of the pin-jointed member i and the direction cosine c_{ij} of the six pin-jointed members and the forces in those members N_i, then according to (5.5) and (5.6) the following applies:

$$\begin{bmatrix} c_{1x} & c_{2x} & \cdots & c_{6x} \\ c_{1y} & c_{2y} & \cdots & c_{6y} \\ c_{1z} & c_{2z} & \cdots & c_{6z} \\ r_{1y}c_{1z} - r_{1z}c_{1y} & r_{2y}c_{2z} - r_{2z}c_{2y} & \cdots & r_{6y}c_{6z} - r_{6z}c_{6y} \\ r_{1z}c_{1x} - r_{1x}c_{1z} & r_{2z}c_{2x} - r_{2x}c_{2z} & \cdots & r_{6z}c_{6x} - r_{6x}c_{6z} \\ r_{1x}c_{1y} - r_{1y}c_{1x} & r_{2x}c_{2y} - r_{2y}c_{2x} & \cdots & r_{6x}c_{6y} - r_{6y}c_{6x} \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \end{Bmatrix} = \begin{Bmatrix} R_x \\ R_y \\ R_z \\ M_{\mathrm{O}x} \\ M_{\mathrm{O}y} \\ M_{\mathrm{O}z} \end{Bmatrix} \tag{5.9}$$

In order that (5.9) can have a solution $\boldsymbol{N}$ for any force-couple system, the matrix on the left must be invertible, i. e. its determinant may not be zero.

Let us select the origin of coordinates O for the coplanar case (three bars in the xz plane) with R_x, R_z, $M_{\mathrm{O}y}$ to be the intersection of bars 1 and 2 ($\boldsymbol{r}_1 = \boldsymbol{r}_2 = \boldsymbol{0}$) and consider the point of pin-jointed member axis 3 on the z axis ($r_{3x} = 0$, $r_{3z} \neq 0$). In this case, removing the second, fourth and sixth rows as well as the fourth, fifth and sixth columns from the matrix in (5.9) gives us the following matrix:

$$\begin{bmatrix} c_{1x} & c_{2x} & c_{3x} \\ c_{1z} & c_{2z} & c_{3z} \\ 0 & 0 & r_{3z}c_{3x} \end{bmatrix} \tag{5.10}$$

and hence

$$\det = r_{3z}c_{3x}(c_{1x}c_{2z} - c_{1z}c_{2x}) \neq 0 \tag{5.11}$$

must apply. Without restricting the universal applicability, it is possible to place the x axis in the direction of bar axis 1, i. e. $c_{1x} = 1$, $c_{1z} = 0$. Consequently, $c_{3x} \neq 0$, $c_{2z} \neq 0$ must be true, i. e. bar axis 3 may not pass through O and bars 1 and 2 may not be collinear. The above requirements are therefore confirmed.

It is often not possible to idealise supports as fully restrained, as has been assumed up to now; instead, it is necessary to consider their flexibility. To this end, in accordance with Fig. 5.10, we use appropriate *translational* and *rotational springs* and in the simplest case presume a linear relationship between the components of the support force-couple system and the corresponding displacements and rotations:

$$A_x = -k_x u_\mathrm{A} \quad , \quad A_z = -k_z w_\mathrm{A} \quad , \quad M_\mathrm{A} = -k_y \varphi_{y\mathrm{A}} \tag{5.12}$$

where k_x, k_z and k_y denote the *stiffnesses of the translational* and *rotational springs*.

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