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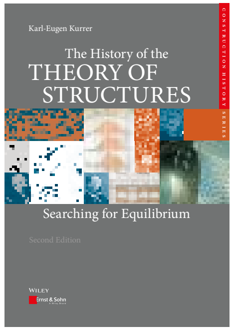
Karl-Eugen Kurrer

The History of the Theory of Structures

Searching for Equilibrium

- **unverzichtbare Hilfe in der Planungspraxis**
- **normenähnlicher Charakter**
- **die Empfehlungen werden auch in Ausschreibungen und Abrechnungen verwendet**

Wann setzte das statische Rechnen im Entwurfsprozess ein? Beginnend mit den Festigkeitsbetrachtungen von Leonardo und Galilei wird die Herausbildung baustatischer Verfahren vorgestellt. Neu in der 2. Aufl.: Erddrucktheorie, Schalentheorie, Computerstatik, FEM, 260 Kurzbiografien.



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ÜBER DAS BUCH

Zehn Jahre nach der 1. Auflage in englischer Sprache legt der Autor sein Buch The History of the Theory of Structures in wesentlich erweiterter Form vor, nunmehr mit dem Untertitel Searching for Equilibrium. Mit dem vorliegenden Buch lädt der Verfasser seine Leser zur Suche nach dem Gleichgewicht von Tragwerken auf Zeitreisen ein. Die Zeitreisen setzen mit der Entstehung der Statik und Festigkeitslehre eines Leonardo und Galilei ein und erreichen ihren ersten Höhepunkt mit den baustatischen Theorien über den Balken, Erddruck und das Gewölbe von Coulomb am Ende des 18. Jahrhunderts. Im folgenden Jahrhundert formiert sich die Baustatik mit Navier, Culmann, Maxwell, Rankine, Mohr, Castigliano und Müller-Breslau zu einer technikwissenschaftlichen Grundlagendisziplin, die im 20. Jahrhundert in Gestalt der modernen Strukturmechanik bei der Herausbildung der konstruktiven Sprache des Stahl-, Stahlbeton-, Flugzeug-, Automobil- und des Schiffbaus eine tragende Rolle spielt. Dabei setzt der Autor den inhaltlichen Schwerpunkt auf die Formierung und Entwicklung moderner numerischer Ingenieurmethoden wie der Finite-Elemente-Methode und beschreibt ihre disziplinäre Integration in der Computational Mechanics. Kurze, durch historische Skizzen unterstützte Einblicke in gängige Berechnungsverfahren erleichtern den Zugang zur Geschichte der Strukturmechanik und Erddrucktheorie vom heutigen Stand der Ingenieurpraxis und stellen einen auch einen wichtigen Beitrag zur Ingenieurpädagogik dar.

Dem Autor gelingt es, die Unterschiedlichkeit der Akteure hinsichtlich ihres technisch-wissenschaftlichen Profils und ihrer Persönlichkeit plastisch zu schildern und das Verständnis für den gesellschaftlichen Kontext zu erzeugen. So werden in 260 Kurzbiografien die subjektive Dimension der Baustatik und der Strukturmechanik von der frühen Neuzeit bis heute entfaltet. Dabei werden die wesentlichen Beiträge der Protagonisten der Baustatik besprochen und in die nachfolgende Bibliografie integriert. Berücksichtigt wurden nicht nur Bauingenieure und Architekten, sondern auch Mathematiker, Physiker, Maschinenbauer sowie Flugzeug- und Schiffbauer. Das vorliegende Buch ist die erste zusammenfassende historische Gesamtdarstellung der Baustatik vom 16. Jahrhundert bis heute.

BESTELLUNG

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Foreword of the series editors

Construction history has experienced amazing momentum over the past decades. It has become a highly vibrant, independent discipline attracting much attention through its international networks. Although research projects at national level focus on different themes, they are united through the knowledge that their diversity in terms of content and methods, and hence the associated synthesizing potential, are precisely the strengths that shape this new field of research. Construction history opens up new ways of understanding construction between engineering and architecture, between the history of building and history of art, between the history of technology and history of science. Since the appearance of the first German edition in 2002, *The History of the Theory of Structures* has become a standard work of reference for this latter field. It continues the series of great works on the history of civil and structural engineering by S. P. Timoshenko and I. Szabó right up to E. Benvenuto and J. Heyman, and enriches them by adding valuable new levels of interpretation and knowledge. We are delighted to be able to publish the second, considerably enlarged, English-language edition as part of the *Construction History Series/ Edition Bautechnikgeschichte*.

Werner Lorenz and *Karl-Eugen Kurrer*
Series editors

Foreword

Ten years after the first English edition of Dr. Kurrer's *The History of the Theory of Structures*, he now presents us with a much enlarged edition, and with a new subtitle: *Searching for Equilibrium* – an addition that reminds us of that most important of all mechanical principles: no equilibrium, no loadbearing system! But the subtitle also expresses the constant search for a balance between theory of structures as a scientific discipline and its prime task in practical applications – totally in keeping with Leibniz' *Theoria cum Praxi*. This interaction has proved beneficial for both sides at all times in history, and runs like a thread through the entire book.

New content in this second edition includes: earth pressure theory, ultimate load method, an analysis of historical textbooks, steel bridges, lightweight construction, plate and shell theory, computational statics, Green's functions, computer-assisted graphical analysis and historical engineering science. Furthermore, the number of brief biographies has been increased from 175 to 260! Compared with the first English edition, the number of printed pages has increased by 50 % to a little over 1,200.

Right at the start we learn that the first conference on the history of theory of structures took place in Madrid in 2005. This theme, its parts dealt with many times, is simply crying out for a comprehensive treatment. However, this book is not a history book in which the contributions of our predecessors to this theme are listed chronologically and described systematically. No, this is 'Kurrer's History of Theory of Structures' with his interpretations and classifications; luckily – because that makes it an exciting journey through time, with highly subjective impressions, more thematic and only roughly chronological, and with a liking for scientific theory. Indeed, a description of the evolution of an important fundamental engineering science discipline with its many facets in teaching, research and, first and foremost, practice.

And what is "theory of structures" anyway? … Gerstner's first book dating from 1789 talks about the "statics of architecture" and Emil Winkler used the term "statics of structures" around 1880. Winkler's term also included earth pressure theory, the evolution of which from 1700 to the present day is now the topic of a new chapter 5 in this second edition.

The history of theory of structures is in the first place the history of mechanics and mathematics, which in earlier centuries were most definitely understood to be applied sciences. Dr. Kurrer calls this period from 1575 to 1825 the "preparatory period" – times in which structural design was still very much dominated by empirical methods. Nevertheless, it is worth noting that the foundations of many structural theories were laid in this period. It is generally accepted that the structural report for the repairs to the dome of St. Peter's in Rome (1742/1743) by the *tre mattematici* represents the first structural calculations as we understand them today. In other words, dealing with a constructional task by the application of scientific methods – accompanied, characteristically, by the eternal dispute between theory and practice (see section 13.2.5). These days, the centuries-old process of the theoretical abstraction of natural and technical processes in almost all scientific disciplines is called 'modelling and simulation' – as though it had first been introduced with the invention of the computer and the world of IT, whereas, in truth, it has long since been the driving force behind humankind's ideas and actions. Mapping the load-bearing properties of building structures in a theoretical model is a typical case. Classic examples are the development of masonry and elastic arch theories (see chapter 4) and the continuum mechanics models of earth pressure of Rankine and Boussinesq (see sections 5.4 and 5.5). It has become customary to add the term 'computational' to these computer-oriented fields in the individual sciences, in this case 'computational mechanics'.

The year 1825 has been fittingly chosen as the starting point of the discipline-formation period in theory of structures (see chapter 7). Theory of structures is not just the solving of an equilibrium problem, not just a computational process. Navier, whose importance as a mechanics theorist we still acknowledge today in the names of numerous theories (Navier stress distribution, Navier-Lamé and Navier-Stokes equations, etc.), was very definitely a practitioner. In his position as professor for applied mechanics at the École des Ponts et Chaussées, it was he who combined the subjects of applied mechanics and strength of materials in order to apply them to the practical tasks of building. For example, in his *Mechanik der Baukunst* of 1826, he describes the work of engineers thus: "... after the works have been designed and drawn, [they] investigate them to see if all conditions have been satisfied and improve their design until this is the case. Economy is one of the most important conditions here; stability and durability are no less important …" (see section 2.1.2.1). Navier was the first to establish theory of structures as an independent scientific discipline. Important structural theories and methods of calculation would be devised in the following years, linked with names such as Clapeyron, Lamé, Saint-Venant, Rankine, Maxwell, Cremona, Castigliano, Mohr and Winkler, to name but a few. The graphical statics of Culmann and its gradual development into graphical analysis are milestones in the history of theory of structures.

Already at this juncture, it is worth pointing out that the development did not always proceed smoothly – controversies concerning the content of theories, or competition between disciplines, or priority disputes raised their heads along the way. This exciting theme is explored in detail in chapter 13 by way of 13 examples.

In the following decades, the evolution of methods in theory of structures became strongly associated with specific structural systems and hence, quite naturally, with the building materials employed, such as iron (steel) and later reinforced concrete (see chapters 8, 9 and 10). Independent materials-specific systems and methods were devised. Expressed in simple terms, structural steelwork, owing to its modularity and the fabrication methods, initially concentrated on assemblies of linear members, not embracing plate and shell structures until the 1950s. On the other hand, reinforced concrete preferred its own two-dimensional design language, which manifested itself in slabs, plates and shells. Therefore, chapters 8 and 10 in this second English edition have been considerably enlarged by the addition of plate and shell structures. The space frames dealt with in chapter 9 represent a link to some extent. This materials-based split was also reflected in the teaching of theory of structures in the form of separate studies. It was not until many years later that the parts were brought together in a homogeneous theory of structures, albeit frequently 'neutralised', i. e. no longer related to the specific properties of the particular building material – an approach that must be criticised in retrospect. Of course, the methods of structural analysis can encompass any material in principle, but in a specific case they must take account of the particular characteristics of the material.

Dr. Kurrer places the transition from the discipline-formation period – with its great successes in the shape of graphical statics and the systematic approach to methods of calculation in member analysis in the form of the force method – to the consolidation period around 1900. This latter period, which lasted until 1950, is characterised by refinements and extensions, e. g. a growing interest in plate and shell structures and the consideration of non-linear effects. Only after this does the 'modern' age of theory of structures begin – designated the integration period in this instance and typified by the use of modern computers and powerful numerical methods. Theory of structures is integrated into the structural planning process of draft design – analysis – detailed design – construction in this period. Have we reached the end of the evolutionary road? Does this development mean that theory of structures, as an independent engineering science, is losing its profile and its justification? The tendencies of recent years indicate the opposite.

The story of yesterday and today is also the story of tomorrow. In the world of data processing and information technology, theory of structures has undergone rapid progress in conjunction with numerous paradigm changes. It is no longer the calculation process and method issues, but rather principles, modelling, realism, quality assurance and many other aspects that form the focus of our attention. The remit includes dynamics alongside statics; in terms of the role they play, plate and shell structures are almost equal to trusses, and taking account of true material behaviour is obligatory these days. During its history so far, theory of structures was always the trademark of structural engineering; it was never the discipline of 'number crunchers', even if this was and still is occasionally proclaimed as such when launching relevant computer programs. Theory of structures continues to play an important mediating role between mechanics on the one side and the draft and detailed design subjects on the other side in teaching, research and practice. Statics and dynamics have in the meantime advanced to what is known internationally as 'computational structural mechanics', a modern application-related structural mechanics.

The author takes stock of this important development in chapters 11 and 12. He mentions the considerable rationalisation and formalisation – the foundations for the subsequent automation. It was no surprise when, as early as the 1930s, the structural engineer Konrad Zuse began to develop the first computer (see section 11.4). However, the rapid development of numerical methods for structural calculations in later years could not be envisaged at that time. J. H. Argyris, one of the founding fathers of the modern finite element method, recognised this at an early stage in his visionary remark "the computer shapes the theory" (1965): Besides theory and experimentation, there is a new pillar – numerical simulation (see section 12.1).

By their very nature, computers and programs have revolutionised the work of the structural engineer. Have we not finally reached the stage where we are liberated from the craftsman-like, formula-based business so that we can concentrate on the essentials? The role of modern theory of structures is discussed in section 14.1, also in the context of the relationship between the structural engineer and the architect. A new graphical statics has appeared, not in the sense of the automation and visual presentation of Culmann's graphical statics, but rather in the form of graphic displays and animated simulations of mechanical relationships and processes. This is a decisive step towards the evolution of structures and to loadbearing structure synthesis, to a new way of teaching structural engineering (see section 14.1.4). This potential as a living interpretation and design tool has not yet been fully exploited. It is also worth mentioning that the boundaries to the other construction engineering disciplines (mechanical engineering, automotive engineering, shipbuilding, aerospace, biomechanics) are becoming more and more blurred in the field of computational mechanics; the relevant conferences no longer make any distinctions. The concepts, methods and tools are universal. And we are witnessing similar developments in teaching, too. No wonder Dr. Kurrer also refers to leading figures from these disciplines. That fact becomes particularly clear in chapter 15, which contains 260 brief biographies of persons who have featured prominently in the theory of structures.

In terms of quality and quantity, this second English edition of *The History of the Theory of Structures* goes way beyond the first edition. This book could only have been written by an expert, an engineer who knows the discipline inside out. Engineering scientists getting to grips with their own history so intensely is a rare thing. But this is one such lucky instance. We should be very grateful to Dr.-Ing. Dr.-Ing. E. h. Karl-Eugen Kurrer, and also 'his' publisher, Ernst & Sohn (John Wiley & Sons), for his *magnum opus*.

Stuttgart, February 2018
Ekkehard Ramm, University of Stuttgart

Preface to the second English edition

Encouraged by the positive feedback from the engineering world regarding the first German edition of my *Geschichte der Baustatik* (2002) and the first English edition *The History of the Theory of Structures* (2008), two years ago I set myself the task of revising my manuscripts, adding new material once again and bringing everything up to date. Increasing the number of pages by a little over 50% was unavoidable, because my goal now was to present a total picture of the evolution of the theory of structures.

But that goal did not just consist of including the research findings of the past few years. Instead, I would now be devoting more space to a detailed treatment of the development of modern numerical methods of structural analysis and structural mechanics as well as the connection between the formation of structural analysis theories and constructional-technical progress. It is for this reason that, for example, plate, shell and stability theories have been paid particular attention, as these theories played an important part in the development of the design languages of steel, reinforced concrete, aircraft, vehicles and ships. As a result, the chapters on steel (chapter 8) and reinforced concrete (chapter 10) have been greatly enlarged. Without doubt, the finite element method (FEM), spawned by structural mechanics and numerical mathematics, was the most important intellectual technology of the second half of the 20th century. Therefore, the historico-logical sources of computational statics plus their development and establishment are now presented in detail separately in chapter 12. Also new is the substantial chapter on the 300-year-old history of earth pressure theory (chapter 5). Earth pressure theory was the first genuine engineering science theory that shaped the scientific self-conception of modern civil engineering, a profession that was beginning to emerge in 18th-century France. It is the reference theory for this profession, and not beam theory, as is often assumed. Not until the 20th century did earth pressure theory gradually become divorced from theory of structures. As in earth pressure theory, it is the search for equilibrium that grabs our historico-logical attention in masonry arch theory. Chapter 4, "From masonry arch to elastic arch", has therefore been expanded. The same is true for chapter 3, which covers the development of theory of structures and applied mechanics as the first fundamental engineering science disciplines. That chapter not only contains the first analysis of textbooks on these two sciences published in the 19th and 20th centuries, but also attempts to extract the scientific and epistemological characteristics of theory of structures and applied mechanics. That therefore also forms the starting point for chapter 14, "Perspectives for a historical theory of structures", the integral constituent of my concept for a historical engineering science, which is explained in detail in this book. Current research into graphical statics is one example mentioned in this chapter, which I summarise under the term "computer-aided graphic statics" (CAGS). The number of brief biographies of the protagonists of theory of structures and structural mechanics has increased by 85 to 260, and the bibliography also contains many new additions.

Probably the greatest pleasure during the preparation of this book was experiencing the support that my many friends and colleagues afforded me. I would therefore like to thank: Katherine Alben (Niskayuna, N. Y.), William Baker (Chicago), Ivan Baláž (Bratislava), Jennifer Beal (Chichester), Norbert Becker (Stuttgart), Antonio Becchi (Berlin), Alexandra R. Brown (Hoboken), José Calavera (Madrid), Christopher R. Calladine (Cambridge, UK), Kostas Chatzis (Paris), Mike Chrimes (London), Ilhan Citak (Lehigh), Zbigniew Cywiński (Gdańsk), René de Borst (Delft), Giovanni Di Pasquale (Florence), Cengiz Dicleli (Constance), Werner Dirschmid (Ingolstadt), Albert Duda (Berlin), Holger Eggemann (Brühl), Bernard Espion (Brussels), Jorun Fahle (Gothenburg), Amy Flessert (Minneapolis), Hubert Flomenhoft (Palm Beach Gardens), Peter Groth (Pfullingen), Carl-Eric Hagentoft (Gothenburg), Friedel Hartmann (Kassel), Hans-Joachim Haubold (Darmstadt), Eva Haubold-Marguerre (Darmstadt), Torsten Hoffmeister (Berlin), Santiago Huerta (Madrid), Peter Jahn (Kassel), Andreas Kahlow (Potsdam), Christiane Kaiser (Potsdam), Sándor Kaliszky (Budapest), Andreas Kirchner (Würzburg), Klaus Knothe (Berlin), Winfried B. Krätzig (Bochum), Arnold Krawietz (Berlin), Eike Lehmann (Lübeck), Werner Lorenz (Cottbus/Berlin), Andreas Luetjen (Braunschweig), Stephan Luther (Chemnitz), René Maquoi (Liège), William J. Maher (Urbana), Gleb Mikhailov (Moscow), Juliane Mikoletzky (Vienna), Klaus Nippert (Karlsruhe), John Ochsendorf (Cambridge, Mass.), Eberhard Pelke (Mainz), Christian Petersen (Ottobrunn), Ines Prokop (Berlin), Frank Purtak (Dresden), Ekkehard Ramm (Stuttgart), Patricia Radelet-de Grave (Louvain-la-Neuve), Anette Rühlmann (London), Jan Peter Schäfermeyer (Berlin), Lutz Schöne (Rosenheim), Sabine Schroyen (Düsseldorf), Luigi Sorrentino (Rome), Valery T. Troshchenko (Kiev), Stephanie Van de Voorde (Brussels), Volker Wetzk (Cottbus), Jutta Wiese (Dresden), Erwin Wodarczak (Vancouver) and Ine Wouters (Brussels).

I am indebted to the technical and design skills of Sophie Bleifuß (typodesign), Siegmar Hiller (production), Uta-Beate Mutz (typesetting) and Peter Palm (drawings), who together helped to guarantee a high-quality production. And without the great support of my family, this book would have been impossible. My dear wife and editor, Claudia Ozimek, initiated the project at the Ernst & Sohn publishing house, and it was my colleague Ute-Marlen Günther who steered the project safely to a successful conclusion. Finally, I would like to thank all my colleagues at Ernst & Sohn who have supported this project and who are involved in the distribution of my book.

I hope that you, dear reader, will be able to absorb the knowledge laid out in this book and not only benefit from it, but also simply enjoy the learning experience.

Berlin, March 2018
Karl-Eugen Kurrer

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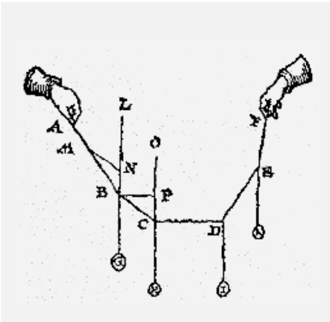
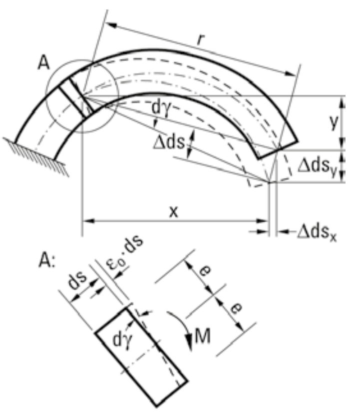


Chapter 4

From masonry arch to elastic arch



The masonry arch is still one of the mysteries of architecture. Anybody who looks into the history of theory of structures quickly encounters this puzzle, the solution to which has occupied countless numbers of scientists and engineers right up to the present day. Since completing his diploma at the Faculty of Theory of Structures at Berlin Technical University in 1981, the author can be counted as belonging to that group. Those studies introduced him to Jacques Heyman's work on the history of theory, which the latter developed into his masonry arch model based on ultimate load theory. A lecture given at the Faculty of Civil Engineering at Stuttgart University, instigated by Prof. Ekkehard Ramm, resulted in a work summarising the development of masonry arch theories since Leonardo da Vinci – and forms the crux of this chapter. Section 4.2.1 was written by Andreas Kahlow and section 4.2.2 by Holger Falter; new findings have found their way into both these sections, one example being the dissertation by Christiane Kaiser. The author would like to take this opportunity to thank Andreas Kahlow and Holger Falter for their kind permission to reproduce their work in this book. The excellent researches of Antonio Becchi, Federico Foce and Santiago Huerta contributed to the success of sections 4.3.1, 4.4.1, 4.4.7 and 4.7; friendships grew out of our many years of cooperation in the field of the history of construction. Numerous ideas resulted from the research of Stefan M. Holzer in the area of the structural assessment of arch structures. The author's dream of a theory of structures within the framework of a historical engineering science took shape through the works of the aforementioned researchers.

The History of the Theory of Structures. Searching for Equilibrium. Second Edition. Karl-Eugen Kurrer.

Jakob Grimm (1785–1863) and Wilhelm Grimm (1786–1859) describe the German noun *Bogen* (= bow, curve, arch) as "… that which is curved, is becoming curved, is rising in a curve" [Grimm, 1860, p. 91], the roots of which lie in the German verb *biegen* (= to bend). A bow (i. e. arch, from *arcus*, the Latin word for arc, bow) in the structural sense is consequently a concave loadbearing structure whose load-carrying mechanism is achieved by way of rigid building materials such as timber, steel and reinforced concrete. When loading such a curved loadbearing structure, a non-negligible part of the external work is converted into internal bending work. Therefore, in German the verb *biegen* not only constitutes the etymological foundation for the noun *Bogen*, but also characterises the curved loadbearing structure from the point of view of the load-carrying mechanism in a very visual and memorable way.

The genesis of the German noun *Gewölbe* (= vault, from *voluta*, the Latin word for roll, turn) is much more complex. Its roots are to be found in Roman stone buildings, as opposed to timber buildings, and in particular the Roman camera, i. e. initially the arched or vaulted ceiling or chamber: "Actually only the word for the curved ceiling, … 'camera' gradually became the term for the whole room below the ceiling. And it is this shift in meaning, which is repeated similarly in *'Gewölbe'*, that leads to the majority of uses for which the latter is regarded as characteristic" [Grimm, 1973, p. 6646].

It is in the German building terminology of the 18th century that we first see the word *Gewölbe* being used in its two-dimensional meaning, whereupon the three-dimensional sense was quickly forgotten. The reason for this may well have been the masonry arch theories that began to surface in the century of the Enlightenment, which started the transitions from loadbearing structure to loadbearing system as a masonry arch model abstracted from the point of view of the loadbearing function – and therefore permitted a quantitative assessment of the load-carrying mechanism in the arch. The beam theory that began with Galileo acted as complement to this terminological refinement. In Zedler's *Universal-Lexikon* dating from 1735, for example, *Gewölbe* is defined totally in the two-dimensional sense, "a curved stone ceiling" [Zedler, 1735, p. 1393], and is differentiated from the suspended timber floor subjected to bending. In 1857 Ersch and Gruber expanded the definition on the basis of the two-dimensional term by mentioning, in addition to dressed stones and bricks, rubble stone material (with mortar joints) as a building material for vaults and arches [Ersch & Gruber, 1857, p. 129]. This became apparent in the material homogenisation of the masonry arch structure that began around 1850 in France, which, in the shape of the plain and reinforced concrete structures of the final decades of the 19th century, paved the way – in the construction sense – for the transition from the theories linked with the materials of the loadbearing masonry arch to the elastic masonry arch theories of Saavedra (1860), Rankine (1862), Perrodil (1872, 1876, 1879, 1880 & 1882), Castigliano (1879), Winkler (1879/1880) and others, and from there to elastic arch theory. The logical nucleus of this historical process is

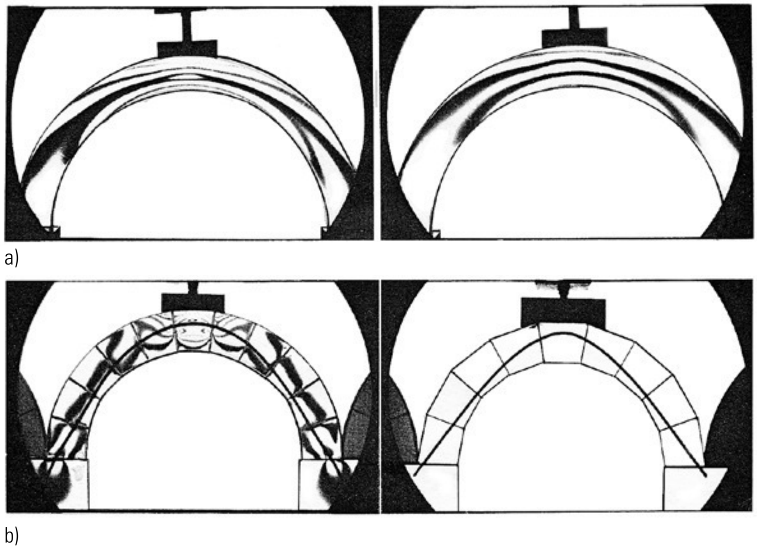


FIGURE 4-1
Photoelastic experiment carried out on a model subjected to a central point load: isochromatic lines of a) monolithic arch model and b) masonry arch model [Heinrich, 1979, pp. 37–38]

the transition from the loadbearing system to the structural system of the elastic arch, e.g. as a concave elastic bar fixed at the abutments. Another thread in elastic arch theory leads us back to the history of timber structures, which Holzer has pursued in two remarkable essays [Holzer, 2007 & 2010/2].

The German noun *Gewölbe* is still used to form compound designations for a number of arch structures, e.g. *Stahlgewölbe and Stahlbetongewölbe* [Badr, 1962, p. 43ff.] (steel and reinforced concrete arches respectively). This contradicts the view that such loadbearing structures work not only in compression, but also in bending as linear-elastic, concave continua. The photoelastic experiments of Bert Heinrich proved the conceptual difference between *Bogen* and *Gewölbe*. Whereas the parallel isochromatic lines in the homogeneous arch indicate high bending stresses (Fig. 4-1a), the loadbearing quality of the (inhomogeneous) masonry arch is characterised purely by the propagation of compression in the direction of the thrust line (Fig. 4-1b).

Summing up, the following definition is proposed: A concave loadbearing structure is a masonry arch when the provision of the loadbearing function is realised solely through rigid building materials with negligible tensile strength which are joined together. Weber has refined this definition and proposed one based on the two-dimensional concept of differential geometry [Weber, 1999, pp. 30–37].

The invention of the masonry arch is, like that of the wheel, impossible to date. In the Berlin Museum of Prehistory & Ancient History, visitors can admire a Mesopotamian burial chamber more than 5,000 years old which is in the form of a barrel vault with a span of a little over 1 m. "False and true arches as used over canals and crypts," writes Ernst Heinrich, "could well date from about the same period even if the one is known to us from the Uruk age, the other from the Mesilim. Both remain … in use until the time of the Seleucids" [Heinrich, 1957–1971, p. 339]. There are without doubt various historico-logical chains of development that culminate in the masonry arch. It is not difficult to imagine that during the construction of a false or corbelled arch the upper stones may have fallen inwards and wedged themselves into an arch shape (Fig. 4-2a), or one or more wedges could have been inserted between two mutually supportive stone slabs to enable the use of shorter slabs (Fig. 4-2b). The same technical motive to reduce the length of a beam and hence increase the bending strength may have encouraged ancient builders to switch from the lintel to the flat arch (see [Huerta, 2012] for the history of the theory of the flat arch) and then to the arch (Fig. 4-2c).

More than 2,000 years certainly passed before the Etruscans' masonry arch with specially cut joints appeared. But the span of time from the first masonry arch theories of the late 17th century to elastic arch theory is less than 200 years. And the analysis of masonry arches based on the ultimate load method did not appear on the scene until the 1960s.

4.1 The arch allegory

Shortly before Christmas 2010, this author received a remarkable letter from Klaus Stiglat [Stiglat, 2010]. The writer of the letter steered the recipient's interest to the arch allegory of the poet Heinrich von Kleist (1777 – 1811) [Földényi, 1999, pp. 161 – 163].

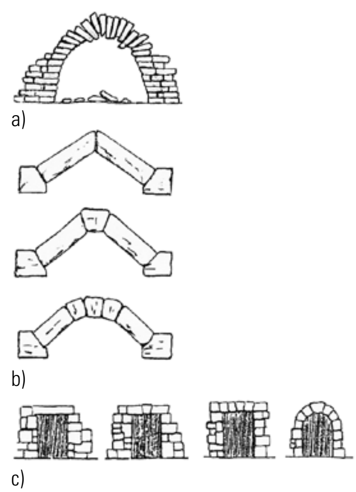


FIGURE 4-2
Historico-logical developments:
a) corbelled arch, b) three-hinge system, and c) from lintel to masonry arch (Heinrich, 1979, pp. 24 – 25)

According to Kleist himself, 16 November 1800 was the "most important day" of his life. As he wrote in a two-part letter to Wilhelmine von Zenge (1780 – 1852) dated 16 and 18 November 1800: "… in Würzburg, I went for a walk. … When the sun went down, it seemed as though my happiness were sinking with it. I was walking back to the city, lost in my own thoughts, through an arched gate. Why, I asked myself, does this arch not collapse, since after all it has *no* support? It remains standing, I answered, *because all the stones tend to collapse at the same time* – and from this thought I derived an indescribably heartening consolation, which stayed by me right up to the decisive moment: I too would not collapse, even if all my support were removed!" (trans. by [Miller, 1982] cited in: [Madsen, 2016, p. 10]). Kleist drew a sketch of the arched gateway in Würzburg and sent it to his "Dear Wilhelmine" on 30 December 1800 (Fig. 4-3).

Kleist's sketch shows seven wedge-shaped stones with the keystone emphasised and a tie that resists the horizontal thrust of the arch. In the ninth scene of his play *Penthesilea* (1808), Prothoë says the following to Penthesilea: "… Stand, stand as does the vaulted arch stand firm, / Because each of its blocks inclines to fall!" (trans. by [Agee, 1998] cited in: [Allen, 2005/2007, pp. 25 – 26]).

In his letter, Klaus Stiglat comments on Kleist's arch allegory thus: "So stability and 'statics' can also be expressed in that way, too – lending humankind stability and 'sanity' " [Stiglat, 2010].

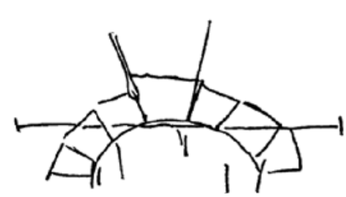


FIGURE 4-3
Kleist's sketch of the arched gateway in Würzburg [Blamberger, 2011, p. 66]

Kleist's image of the lintel as support is more than just the essence of a private theory shared with Wilhelmine von Zenge, as Günter Blamberger writes [Blamberger, 2011, p. 66]. Instead, in the form of the gauged arch, it represents statics as a theory of equilibrium per se – yet announcing the lintel as support through the wedging together of the stones at the historico-logical transition from the false to the true arch (Fig. 4-2a) a completely new type of equilibrium configuration.

4.2 The geometrical thinking behind the theory of masonry arch bridges

Whereas the large bridges of the late Renaissance demonstrated innovations primarily through the use of geometry, the application of the methods of statics in design remained the province of the Baroque.

More precise variation in possible design geometries, the centering, the foundations and the construction sequence, etc. was now feasible through the use of drawings, ever-better dimensional accuracy and precision in the designs. Using the examples of the Ponte S. Trinità in Florence and the Fleisch Bridge in Nuremberg, it will be shown how these new design approaches gradually became accepted in bridge-building.

During the first decades of the 18th century, bridge-building progressed via the intermediate stages of the first attempts to quantify this subject (La Hire, Couplet, Bélidor) to become the number one object of masonry arch theory. The idea of the thrust line became, indirectly, the focus of all deliberations: conceptual designs concerning the functional mechanism of bridges and intensive communication between experts advanced the formulation of bridge-building theories.

4.2.1 The Ponte S. Trinità in Florence

The end of the 16th century marked the start of a new evolutionary era in the building of masonry arch bridges. The Renaissance initially took the structures and forms of construction of the Romans as its models. Owing to its rise/span ratio of 1 : 2, the semicircular arch permits only very restricted functionality and is therefore unsuitable for urban structures in particular. This functional disadvantage gave rise to new arch forms that were considerably shallower than the Roman arch.

Besides longer spans, the rise/span ratio also increased. The classical ratio was around 1 : 3, but in the case of the Ponte Vecchio (5 m rise, 32 m span) by Taddeo Gaddi (1300 – 1366), this increased during the late Middle Ages to 1 : 6.5. However, a new approach to design – and not just spans longer than those of the late Middle Ages – was the main aspect that signalled the leap in quality of the Renaissance compared with ancient

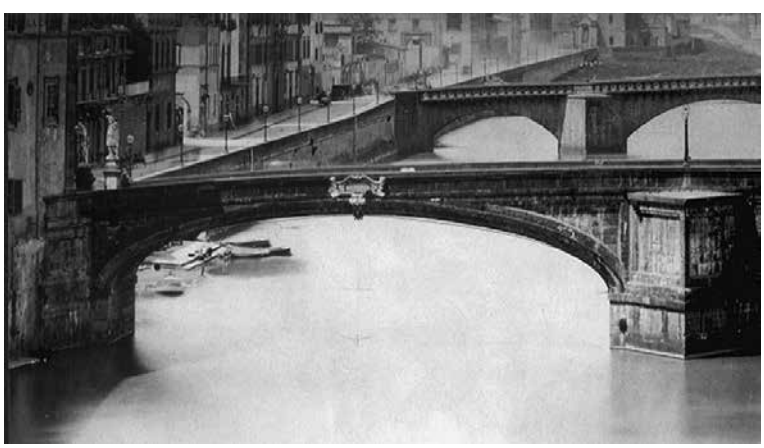
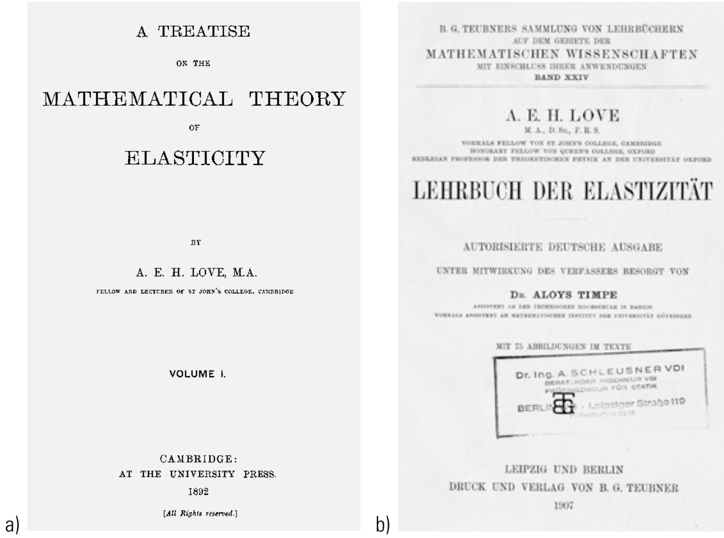


FIGURE 4-4
Ponte S. Trinità, photo taken prior to the bridge's destruction in the Second World War (photo: Gizdulich collection)


A TREATISE
ON THE
MATHEMATICAL THEORY
OF
ELASTICITY
BY
A. E. H. LOVE, M.A.
FELLOW AND LECTURER OF ST JOHN'S COLLEGE, CAMBRIDGE
VOLUME I.
CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1892
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B. G. TEUBNERS SAMMLUNG VON LEHRBÜCHERN
AUF DEM GEBIETE DER
MATHEMATISCHEN WISSENSCHAFTEN
MIT EINSCHLUSS IHRER ANWENDUNGEN
BAND XXIV
A. E. H. LOVE
M. A., D. Sc., F. R. S.
LEHRBUCH DER ELASTIZITÄT
AUTORISIERTE DEUTSCHE AUSGABE
UNTER MITWIRKUNG DES VERFASSERS BESORGT VON
Dr. ALOYS TIMPE
MIT 75 ABBILDUNGEN IM TEXTE
Dr. Ing. A. SCHLEUSNER VDI
LEIPZIG UND BERLIN
DRUCK UND VERLAG VON B. G. TEUBNER
1907

a) b)

FIGURE 10-40
Title pages of a) the first volume [Love, 1892/1893], and b) the German translation of the collected edition [Love, 1907] of Love's *Treatise*

classical phase (1875–1900) and the accumulation phase (1900–1925) of theory of structures that engineers built the golden bridge from mathematical to analytical shell theory.

Practice makes do: from tank formula to tank theory

Practising engineers initially approached shell theory cautiously via the analysis of the simplest shell form, the single-curvature, fixed cylindrical shell; but the representatives of fundamental engineering science disciplines such as applied mechanics and theory of structures were no different (Fig. 10-41). Using this structural model, engineers attempted to size vessels of steel and, later, reinforced concrete – the works of E. Winkler (1860), F. Grashof (1878), G. A. Wayss (1887), V. G. Shukhov (1888) (see [Ramm, 1990]), P. Forchheimer (1894), R. Maillart (1903) (see [Schöne, 1999, 2011]), C. Runge (1904), Panetti (1906), H. Müller-Breslau (1908), H. Reissner (1908), K. Federhofer (1909, 1910), T. Pöschl and K. v. Terzaghi (1913) and A. and L. Föppl (1920) should be mentioned here. In 1923 V. Lewe summarised the methods for the structural calculation of liquid-retaining structures in a longer article for the *Handbuch für Eisenbetonbau* (reinforced concrete manual) [Lewe, 1923].

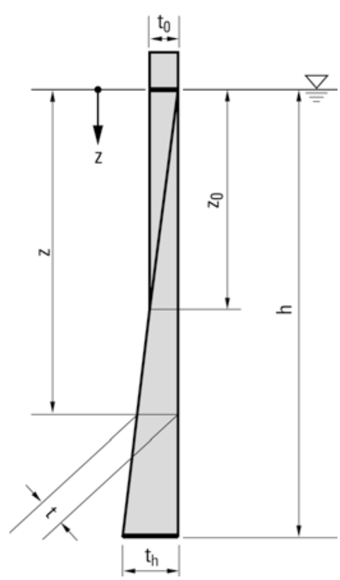
In his *Monier-Broschüre*, G. A. Wayss specifies an equation for determining the wall thickness *t(z)* of a reinforced concrete water tank [Wayss, 1887, p. 34] which he derived from the boiler formula (eq. 8-35) (Fig. 10-41):

$$t(z) = t = r \cdot \frac{p_i}{\sigma_{permiss}} = r \cdot \frac{\gamma \cdot z}{\left[\sigma_{b,permiss} + \frac{1}{n}\left(\sigma_{s,permiss} - \sigma_{b,permiss}\right)\right]} \qquad (10\text{-}55)$$

where:

r	internal radius of water tank
$t(z)$	wall thickness
$\gamma \cdot z$	hydrostatic pressure at depth z below the surface of the water
$\sigma_{b,permiss}$	permissible tensile stress of concrete
$\sigma_{s,permiss}$	permissible tensile stress of steel

FIGURE 10-41
On the design of a reinforced concrete cylindrical water tank with a partially linearly varying wall thickness after Wayss [Wayss, 1887, p. 34]



n ratio of concrete cross-sectional area A_b to steel cross-sectional area A_s (i.e. amount of reinforcement per unit length in *z* direction)

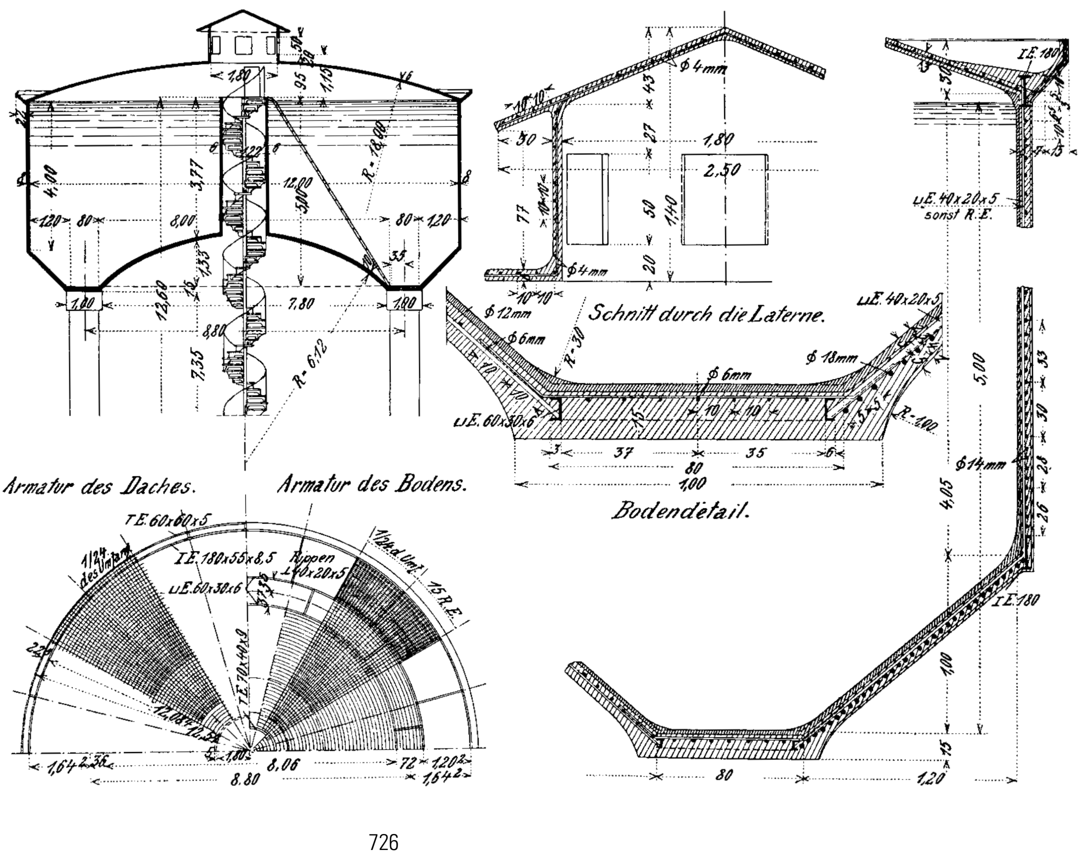
Theoretically, eq. 10-55 should always result in $t(z = 0) = 0$ when $z = 0$, but in practical terms a certain wall thickness t_0 with a steel cross-section $A_{s0} = t_0 / n$ always results. For this reason, Wayss proposed a wall thickness t_0 with a steel-cross-section A_{s0} up to a height $z = z_0$, which, according to eq. 10-55, would produce the value t_0, and only after that would the linear change in wall thickness down to the base of the tank be determined for $z = h$ according to eq. 10-55. For this latter section, Wayss specified a simple construction according to the intercept theorem (see Fig. 10-41):

$$\frac{t}{t_h} = \frac{z}{h} \tag{10-56}$$

Eq. 10-55 only takes into account the hoop tension stresses in the φ direction (see Fig. 10-39b); the normal stresses in the ϑ direction (see Fig. 10-39b) are not entered into the boiler formula.

The building of tanks etc. in reinforced concrete reached a new height after the 1890s. French building contractors became the leaders here, with about 10 companies competing to achieve the best form. In 1898 the company founded by Edmond Coignet (1856–1915) in 1890 set up two identical water tanks with a capacity of 500 m³ and a wall thickness of 8 cm (Fig. 10-42). This structure can be regarded as the prototype for reinforced concrete construction at the transition from the classical phase (1875–1900) to the accumulation phase (1900–1925) of theory of struc-

FIGURE 10-42
General arrangement and reinforcement drawings for the water tank at the Navy arsenal in Toulon [Wuczkowski, 1910, p. 574]



tures; it formed, so to speak, the foundation for the genesis of reinforced concrete shells.

Coignet's monolithic water tank consists of several shells: a domed roof with central lantern light, internal and external cylindrical shells, a truncated cone shell and a domed base ("Intzeboden"). This reinforced concrete structure is supported on a masonry cylinder with 80 cm thick walls. The tank patented by Otto Intze (1843 – 1904) [Olbrisch, 1974] in 1883 is characterised by the fact that the horizontal thrust from the domed base is cancelled out by that from the truncated cone, meaning that the masonry cylinder is subjected to vertical forces only. This type of tank was widely used for storing water for railways, industry and waterworks. Coignet based the design of the two domes on membrane theory and specified rolled sections for their tension rings. For example, the upper tension ring was designed according to the equation

$$\sigma_{s,\,exist} = \frac{H \cdot r}{A_s} \leq \sigma_{s,\,permiss} = 1000 \text{ kg/cm}^2 \qquad (10\text{-}57)$$

where A_s is the steel cross-section, r the radius of the dome on plan (6 m in this case) and H the following horizontal thrust due to membrane theory:

$$H = \frac{p \cdot (R - f)}{R} \qquad (10\text{-}58)$$

with radius of curvature R (18 m in this case), rise of the dome f (0.95 m) and vertical support reaction p per metre due to the self-weight of the shell [Wuczkowski, 1910, pp. 575 – 576]. The derivation of eqs. 10-57 and 10-58 can also be found in the dissertation by Lutz Schöne [Schöne, 2011, pp. 48 – 49]. Coignet was unable to investigate how the membrane stress state is disturbed at the shell transitions, since engineering practice was ahead of theory formation and Coignet devised an elegant construction detail. Schöne carried out a structural analysis of Coignet's water tank in his dissertation and proved that the domes were adequately designed with respect to their static load-carrying capacity. He concludes that "the dome therefore exhibits high redundancy with respect to high loads, imperfect geometries or unintended situations during construction. This was certainly the reason why this type of tank could be built very economically" [Schöne, 2011, Annex 1, p. 9]. In Germany alone, more than 400 "Intze" water tanks were built between 1888 and 1904 [anon., 1905/1, p. 15] – most of them in steel. The economic "Intze" water tanks in reinforced concrete *à la* Coignet were now growing to be a serious rival to the steel tank, as the article by Richard Wuczkowski in the *Handbuch für Eisenbetonbau* (reinforced concrete manual) shows [Wuczkowski, 1910, pp. 574 – 578].

Reinforced concrete also started to be used for the building of gas tanks around the turn of the 20th century. Robert Maillart set a milestone with the two gas tanks built in St. Gallen, Switzerland, in 1902/1903. He was the first to consider the bending stresses due to M_ϑ, which he was able to obtain from an iterative graphical analysis (Fig. 10-43); the deflection curve of the tank wall in the meridional direction was determined with the help of Mohr's analogy. Taking the radius of curvature R of the deflection curve from the graphical analysis, Maillart calculated the bending moment

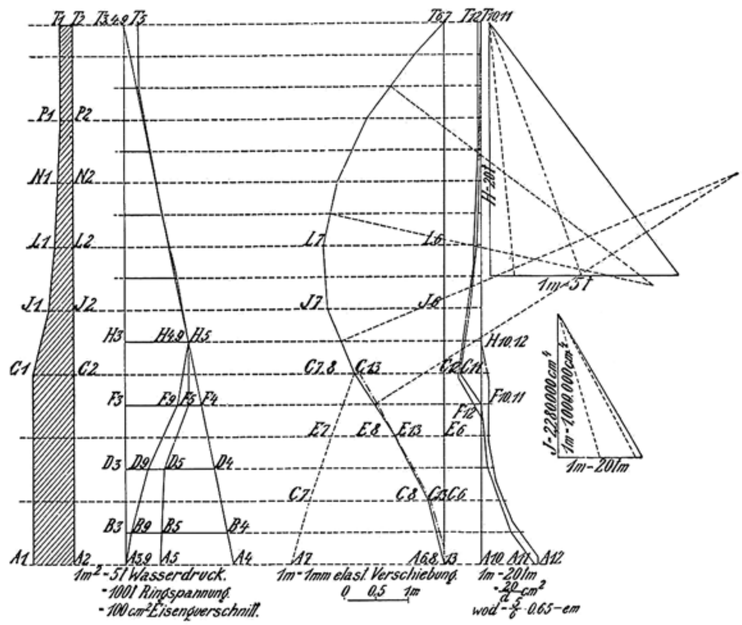


FIGURE 10-43
Graphical analysis of a gas tank by Maillart [Wuczkowski, 1910, p. 485]

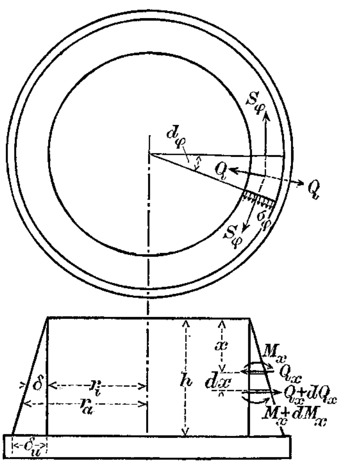


FIGURE 10-44
A cylindrical shell with a linearly varying wall thickness and fixed at the base [Reissner, 1908, p. 150]

diagram using the familiar relationship $M_\vartheta = (E \cdot I)/R$. Lutz Schöne endorsed Maillart's ingenious structural analysis in his dissertation – as Jörg Schlaich says – as "thinking in deformations" [Schöne, 2011, p. 51].

Shortly afterwards, Runge (1856–1927), a mathematician, published an approximation calculation for a cylindrical water tank with stepped wall thicknesses [Runge, 1904]. Picking up on this, Hans Reissner investigated a cylindrical shell with a linearly varying wall thickness (Fig. 10-44), which led to a fourth-order differential equation with varying coefficients with no closed-form solution. Reissner resolved this differential equation using power series and prepared it in the form of tables and charts.

But uncertainties still existed in the structural analysis of tanks etc. and this was expressed in the work of Emil Reich [Reich, 1907]. Reissner criticised not only Reich's awkward solution, but also his sample calculation, which results in a value for the wall thickness seven times the radius of the tank! Following Reissner's work, Federhofer proposed a graphical method for determining the stress distribution in cylindrical tank walls with any wall thickness [Federhofer, 1909, 1910]. The bending theory for cylindrical shells as a practical structural model for reinforced concrete tanks was fully developed by about 1915.

Schwedler's comeback!

The first major step in the direction of a structural membrane theory for shells of revolution was taken by J. W. Schwedler in 1863 and 1866. He realised that in the structural analyses of domes it was not only the meridional stresses σ_ϑ that had to be quantified (as had been the case in the past), but also the hoop stresses σ_φ. Schwedler derived the equilibrium conditions for a dome-type shell of revolution with any geometry (see Fig. 10-39a) and specialised them for shallow shell surfaces and for spherical surfaces [Schwedler, 1863]. In a further paper, he used his structural membrane theory for shells of revolution to calculate the member forces in the space frame he had invented – the Schwedler dome [Schwedler, 1863] (see section 9.1). As the internal forces in the radial and tangential directions of rotationally symmetric membrane shells can be determined from the equilibrium conditions alone, i. e. this is an internally statically determinate system, graphical analysis was already being used to analyse such loadbearing systems in the late 1870s (see section 7.3.5.1). In the *Monier-Broschüre*, Schwedler's membrane theory was used to design dome-type reinforced concrete shells [Wayss, 1887, pp. 31 – 33]. For the dome with radius of curvature R, the meridional stress per unit length of the circumference is

$$\sigma_\vartheta = p \cdot R \ \frac{1}{(1 + cos\,\vartheta)} \tag{10-59}$$

and the hoop stress per unit length of the meridian is

$$\sigma_\varphi = p \cdot R \cdot \left[cos\,\vartheta - \frac{1}{(1 + cos\,\vartheta)}\right] \tag{10-60}$$

[Wayss, 1887, p. 32] (for designations see Fig. 10-39a). In both the above equations p is the weight per unit area of the dome surface including imposed loads, which is imposed in the radial direction equally throughout. Whereas the meridional stresses always lie within the compressive stress range, with a value of $0.5 \cdot p \cdot R$ at the crown and increasing towards the springings, the hoop stress changes its sign at ϑ = 51.83°, i. e. the hoop stresses are compressive at the top and tensile at the bottom. The tensile hoop stresses of a hemispherical dome have the value $\sigma_\varphi = -p \cdot R$ at the base, which is taken as the basis for the design. That results in the following steel reinforcement cross-sections [Wayss, 1887, S. 33]:

In the hoop direction per unit of length of the meridian

$$A_{s,\,\varphi} = \frac{p \cdot R}{\text{zul}\,\sigma_s} \tag{10-61}$$

and in the meridional direction per unit length of the circumference

$$A_{s,\,\vartheta} = \frac{t}{n} \tag{10-62}$$

Eq. 10-62 (where t = shell thickness) is empirical because the denominator is given as $n > 1$. The reinforcement was laid in the radial and tangential directions. Reinforced concrete domes would be calculated according to this method up until the middle of the accumulation phase of theory of structures (1900 – 1925).

As part of his history of construction studies concerning the Bavarian Army Museum (1902 – 1904) and the Anatomical Institute (1905 – 1907) in Munich, Marco Pogacnik discovered the structural calculations for these buildings [Pogacnik, 2009]. Both were built by the Eisenbeton-Gesellschaft, a merger between Wayss & Freytag and Heilmann & Littmann which took place in 1903 with the aim of carrying out reinforced concrete projects in and around Munich. Fig. 10-45 shows the cover to the structural calculations for the dome at the Bavarian Army Museum, which were produced by Heilmann & Littmann.


Armee-Museum
München
Construktion des Kuppelbaues in eisenarmirtem Beton nach System Hennebique.
— Statische Berechnungen —
Zulässige Inanspruchnahme der Materialien
Eisen auf Zug u. Druck 1000 kg/qcm
Eisen auf Abscherung 700 kg/qcm
Beton auf Druck 25 kg/qcm
Nutzlast 250 kg/m² Horizontalfläche
Innere Kuppel u. Dekoration 150 kg/m²

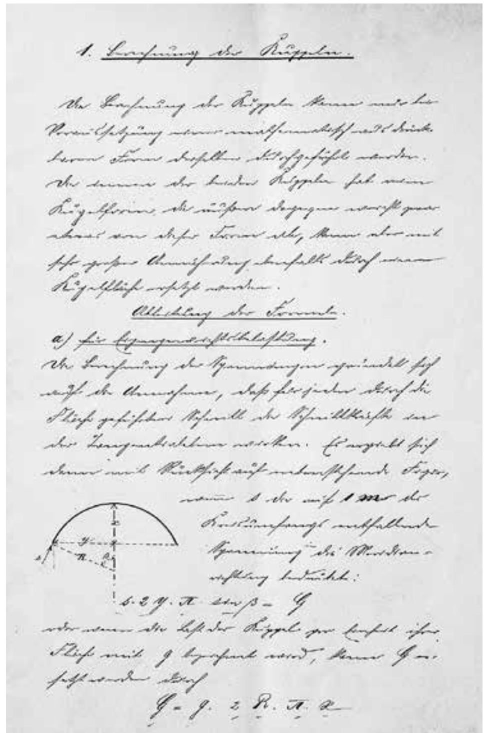


FIGURE 10-45
(above left) Cover to the structural calculations dated 9 February 1903 for the dome at the Bavarian Army Museum, which were produced by Heilmann & Littmann [Pogacnik, 2009, p. 346]

FIGURE 10-46
(above right) Structural calculations by Emil Mörsch dated 15 April 1903 for the dome at the Bavarian Army Museum [Pogacnik, 2009, p. 348]

The inner and outer domes (16 m span) were to be built using the Hennebique system and consisted of ribs in the meridional and circumferential directions, i. e. consisted of curved T-beams. The following permissible stresses were assumed:

- steel in tension and compression: $\sigma_{s, permiss} = 1{,}000$ kg/cm^2
- steel in shear: $\tau_{s, permiss} = 700$ kg/cm^2
- concrete in compression: $\sigma_{b, permiss} = 25$ kg/cm^2

An imposed load $p = 250$ kg/m^2 and self-weight of the inner dome with decoration $g = 150$ kg/m^2 was applied horizontally, resulting in a total load $q = 400$ kg/m^2. However, this design was not built because, shortly before, Wayss & Freytag decided against the Hennebique system owing to the excessive licence fees (see section 10.2.2.3). Instead, Emil Mörsch from Wayss & Freytag submitted 22 pages of structural calculations for a totally new concept with two spherical reinforced concrete shells (Fig. 10-46).

Mörsch applied Schwedler's membrane theory and assigned the forces to the T-section (40 and 45 mm deep) in the meridional and circumferential directions of the 6 cm thick shell. The shells of the Bavarian Army Museum can be interpreted as a further development of the Melan system (see section 10.2.2). Even the bolder, 22 m span, 10 cm thick dome with a rise $f = 5.75$ m is based on the Melan system. "The calculations were carried out according to the method for Schwedler domes for the various load cases during construction and in service" [Siegfried, 1908, p. 148]. Responsible for the calculations dated 17 May 1905 was not Mörsch this time, but Reiner from the Eisenbeton-Gesellschaft [Pogacnik, 2009, p. 352]. So by about 1905, the calculation of reinforced concrete domes according to Schwedler's membrane theory had become established in the practical calculations of reinforced concrete engineers. Nevertheless, the analytical assessment of the transfer of the forces to the supports of the shell was still a closed book.

Theory in practice: the membrane shell of St. Blaise Cathedral

Heinrich Spangenberg (1879 – 1936), director of the Karlsruhe branch of Dyckerhoff & Widmann, working with Otto Mund, designed a pure membrane shell for the St. Blaise abbey church, the so-called St. Blaise Cathedral (Fig. 10-47). The overall structure has a diameter of 33.70 m, and the inner dome in the form of a membrane shell (1910 – 1913) spans 15.40 m with a rise f = 1.49 m and shell thickness t = 8 – 12 cm [Spangenberg, 1912]. This shell, too, was calculated using the Schwedler method. The two engineers supported the shell on 20 radial struts integrated tangentially in the shell, essentially in keeping with the requirements of a membrane. The struts widen, haunch-like, around the edge of the shell so that the meridional stresses are grouped together as normal forces in the radial struts via the arching effect. By contrast, the hoop tensile stresses of the inner dome are carried by a continuous tension ring beam positioned around the edge of the shell. Here, too, the continuity principle for focusing the load

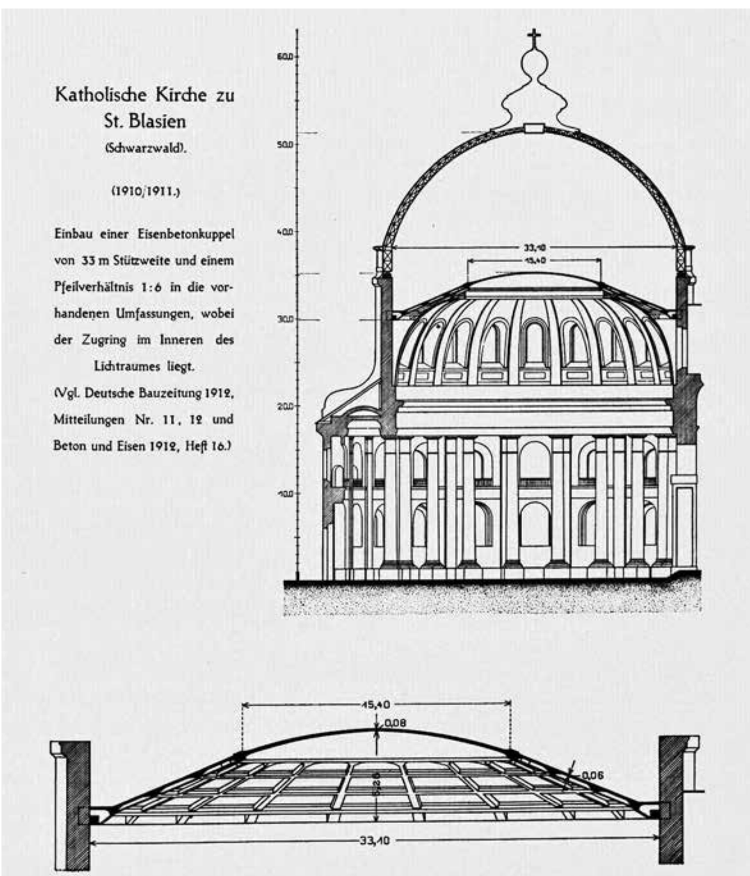


FIGURE 10-47
Inner dome of St. Blaise Cathedral [Dyckerhoff & Widmann, 1920, p. 45]

path was systematically implemented in the actual construction. The four 20-sided ring systems serve to brace the 20 radial struts and were analysed graphically together with the latter. There is a base ring at the springings with encased steel sections that have to carry a hoop tension force of 156 t. Therefore, the existing drum masonry on which the steel outer dome rests remains essentially unaffected by the horizontal forces of the radial struts to the inner dome.

The load-carrying system of the inner dome, which suits the membrane approach, required a complicated three-dimensional system of members. This system with its several degrees of static indeterminacy was analysed by Spangenberg and Mund with the help of the force method. In 1925 Franz Dischinger praised this shell design as "the boldest construction so far" [Dischinger, 1925, p. 362]. Nonetheless, it constituted an erratic element in reinforced concrete shells at that time. "The design was so closely based on a particular interpretation of membrane theory plus the specific conditions and restrictions that it cannot serve as a model for shells for buildings generally. So this approach, too, was only one step on the way to a better understanding of shells" [Schöne, 2011, p. 71].

Bending theory for shells of revolution takes shape

The fact that the membrane stress condition in shells is 'disturbed' by bending stresses at the supports was already well known by the middle of the accumulation phase of theory of structures (1900 – 1925). Fig. 10-48 illustrates this problem at the edge for the simplest case. Owing to the external loads (e. g. self-weight g), an elastic displacement of the dome w_R and a radial displacement of the base ring u_R ensues at the impost joint. As the impost joint may not open (compatibility condition), it must be closed by the meridional bending moment $M_{\vartheta=\alpha}$. The ensuing meridional bending moments M_ϑ decay like attenuated vibrations. It was the quantitative ascertainment of this disturbance at the edge in the form of the decay factor λ that finally led to a structural bending theory for shells.

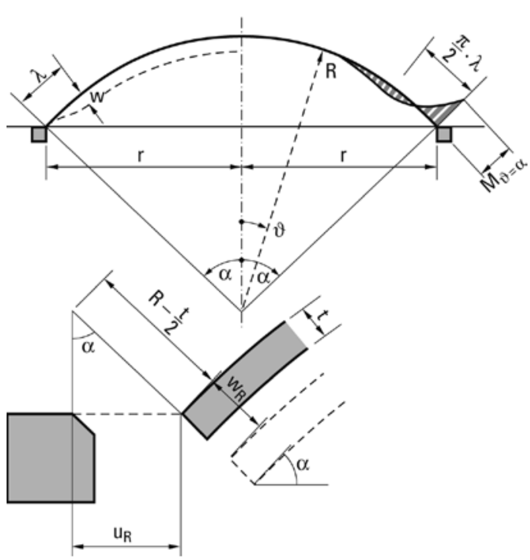


FIGURE 10-48
Disturbed membrane stress state at the edge of a shell with constant thickness

ing theory [Strecker & Feldtkeller, 1929]. In his classic monograph on quadripole theory, Feldtkeller, who was appointed professor of electrical telecommunications technology at Stuttgart Technical University in 1936 following his work in Siemens' central laboratory in Berlin, systematically used the formal potential of matrix calculation for calculating linear electrical networks [Feldtkeller, 1937]. W. Quade finally provided an overview of the most important applications of matrix calculation for electrical networks and vibrations [Quade, 1940]. Feldtkeller's 1937 monograph helped the quadripole theory to become the showcase of matrix calculation in the fundamental engineering science disciplines. Two years later, the electrical engineer G. Kron, an employee of General Electric, published his book entitled *Tensor Analysis of Networks* [Kron, 1939]. Kron unfortunately mixed tensor and matrix theory. So the introduction of matrix calculation into electrical engineering experienced an unlucky start due to a number of less-than-fortunate publications [Zurmühl, 1950, p. 347]. Notwithstanding, Kron was able to cross the boundary between electrical engineering and mechanics. For example, he used the analogy between electrical and mechanical networks (elastic trusses) known to Maxwell and Kirchhoff for analysing three-dimensional trusses and formulated them in the language of matrix theory [Kron, 1944]. Kron's work inspired the aircraft engineer B. Langefors, an employee of the Swedish SAAB company, to summarise the force method in matrix form [Langefors, 1952]. Working independently, H. Falkenheiner published two articles in French [Falkenheiner, 1950, 1951], which Alf Samuelsson compared with the work of Langefors (1952): "The papers by Falkenheiner and Langefors are very similar. Both use the principle of deformation minimum according to Menabrea-Castigliano to deduce the matrix of influence coefficient expressing point displacements as a function of point loads. They also both describe a substructure technique. Langefors uses force in hypothetical cuts as redundants while Falkenheiner uses superposition coefficients of equilibrium systems as redundants. The method of Falkenheiner is then more general than that by Langefors" [Samuelsson, 2002, p. 7]. In 1953 Falkenheiner discussed his two articles in the light of the work of Langefors [Falkenheiner, 1953].

11.5.3 The integration of matrix formulation into engineering mathematics

One of the historical trails of matrix formulation in structural mechanics leads back to the Aerodynamics Department set up in 1925 by R. A. Frazer at the National Physics Laboratory in Teddington near London. Together with W. J. Duncan, Frazer researched the flutter of aircraft wings and in 1928 published the so-called *Flutter Bible* [Felippa, 2001]. Six years later, Duncan and A. R. Collar formulated conservative vibration problems in the language of matrix algebra [Duncan & Collar, 1934], and one year after that wrote a work on the motion equations of damped vibrations with the help of the powerful mathematical resources of matrix algebra [Duncan & Collar, 1935]. Looking back, Collar described this discovery of matrix algebra for a reformulation of vibration mechanics as follows: "Frazer had studied matrices as a branch of applied mathematics under Grace in Cambridge; and he recognized that the statement of, for example, a ternary flutter problem in terms of matrices was neat and compendious. He was, however, more concerned with formal manipulation and transformation to other coordinates than with numerical results. On the other hand, Duncan and I were in search of numerical results for the vibration characteristics of airscrew blades; and we recognized that we could only advance by breaking the blade into, say, 10 segments and treating it as having 10 degrees of freedom. This approach also was more conveniently formulated in matrix terms, and readily expressed numerically. Then we found that if we put an approximate mode into one side of the equation, we calculated a better approximation on the other; and the matrix iteration procedure was born" [Collar, 1978, p. 17]. The year 1938 saw Frazer, Duncan and Collar publish the first monograph in which areas of structural dynamics such as aeroelasticity were formulated systematically in

Take as generalised coordinates q_1, q_2, q_3 the linear displacements of B, F, G, respectively. Then the displacement of D is $\frac{1}{2}(q_1+q_2)$. In a general static displacement of the system, the elastic moments at A, C, E, D, will be $\frac{1}{3}q_1$, $\frac{2}{3}(q_1+q_2)$, $\frac{1}{3}q_2$, $\frac{1}{9}(q_3-q_1-q_2)$, and since the lever arms are all of unit length, the vertical forces at B, D, F, G are also $\frac{1}{3}q_1$, $\frac{2}{3}(q_1+q_2)$, $\frac{1}{3}q_2$, $\frac{1}{9}(q_3-q_1-q_2)$. To find the flexibility matrix, apply unit load at B, F, G in succession. When unit load is applied at B, we have by moments about AE,

$$1 = \tfrac{1}{3}q_1+\tfrac{2}{3}(q_1+q_2)+\tfrac{1}{3}q_2 = q_1+q_2,$$

while by moments about AB,

$$\tfrac{2}{3}q_2+\tfrac{2}{3}(q_1+q_2) = 0, \quad \text{or} \quad q_1+2q_2 = 0.$$

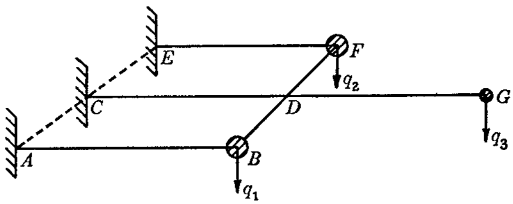


Fig. 10·8·1

Hence $q_1 = 2$, $q_2 = -1$, and since the moment at D is zero, $q_3 = q_1+q_2 = 1$. The displacements are thus $\{2, -1, 1\}$. Similarly, when unit load is applied at F, the displacements are $\{-1, 2, 1\}$. When unit load is applied at G, we have by moments about AE,

$$2 = \tfrac{1}{3}q_1+\tfrac{2}{3}(q_1+q_2)+\tfrac{1}{3}q_2 = q_1+q_2,$$

and, since the displacement is symmetrical, $q_1 = q_2 = 1$. Moreover, by moments about BF,

$$1 = \tfrac{1}{9}(q_3-q_1-q_2) \quad \text{or} \quad q_3 = 11.$$

Hence in this case the displacements are $\{1, 1, 11\}$. The flexibility matrix is thus

$$\Phi = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 11 \end{bmatrix}.$$

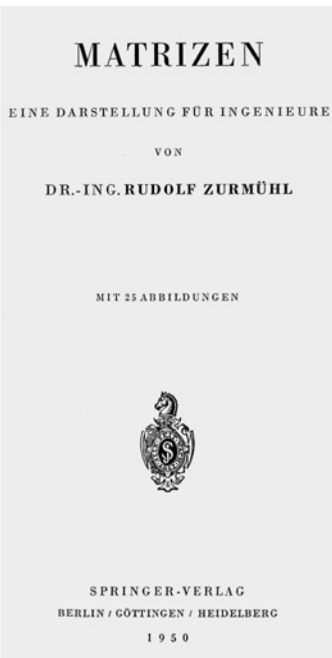
The inertia matrix is evidently

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

FIGURE 11-36
Eigenvalue analysis of a system of bars with three degrees of freedom after Frazer, Duncan and Collar [Frazer et al., 1963, p. 323]

terms of matrix algebra (Fig. 11-36) [Frazer et al., 1938]; since the end of the consolidation period of theory of structures (1900 – 1950), this has become a standard work for engineers who wish to find out something about solving vibration problems using matrices. Fig. 11-36 shows the eigenvalue analysis of a system of bars with the three degrees of freedom q_1, q_2 and q_3, which was investigated with the help of matrices; Fig. 11-36 is taken from the seventh unaltered reprint of the original edition of 1938. The monograph thus remained relevant until the middle of the innovation phase of theory of structures (1950 – 1975).

Zurmühl's monograph *Matrizen. Eine Darstellung für Ingenieure* (matrices – an explanation for engineers) of 1950 (Fig. 11-37) represented a milestone in the use of matrix formulation in the German-speaking countries. He realised that matrix formulation provided linear algebra with a means of expression that could be used to express the linear relationships prevailing in physics and the engineering sciences for operations that were uniform but difficult to present in customary mathematical language through equations of unsurpassed conciseness and clarity that always concentrate the user's attention on the essentials (see [Zurmühl, 1950, p. I]). Matrix theory will "assert itself more and more in engineering mathematics and perhaps soon play a similar role to vector theory, which today is indispensable" [Zurmühl, 1950, p. I]. Zurmühl's vision would very soon become reality as, during the 1950s, his monograph became the standard work on engineering mathematics. The book had been backed up since 1945 by the work of Alwin Walther (1898 – 1967), who tested numerical methods and procured obscure literature. It was at the Institute of Practical Mathematics (IPM), headed by Walther, at Darmstadt Technical University that Zurmühl investigated a matrix-based iteration method in the early 1940s, which he tested using the example of the calculations for a three-dimensional trussed framework with multiple degrees of static indeterminacy (see [Zurmühl, 1950, p. 282]).


MATRIZEN

EINE DARSTELLUNG FÜR INGENIEURE

VON

DR.-ING. RUDOLF ZURMÜHL

MIT 25 ABBILDUNGEN

SPRINGER-VERLAG
BERLIN / GÖTTINGEN / HEIDELBERG
1950

FIGURE 11-37
Title page of the first German book on the application of matrices to engineering and the engineering sciences

Even before the Second World War, Walther's IPM was being called a "computations factory", and in 1939 up to 70 female workers equipped with mechanical tabletop calculating machines were performing tasks associated with ballistics, lightweight construction, radiolocation and optics (see [Petzold, 1992, p. 226]). The thinking work of engineering science calculation had thus been schematised and divorced completely from the engineering work. What could have been more obvious than to automate this calculation work, as Zuse had suggested back in 1936?

Plans for a large, powerful, automatic program-controlled computing installation, which was to be assembled from parts for current calculating machines, were therefore discussed as early as 1943 at the IPM, which Walther had made available for research into wartime issues. Spurred on by the message concerning Aiken's large Mark I Automatic Sequence Controlled Calculator (ASSC), the generals of the German armed forces allocated the highest priority to Walther's project, which meant that he could procure the parts he needed to assemble the machine within a very short time. But a few days later the new installation disappeared into the bombed-out ruins of the IPM (see [Petzold, 1992, p. 228]). Through Prof. Herbert Wagner, manager of Special Department F at Henschel-Flugzeugwerke AG and, as such, Zuse's superior (Zuse had headed the structural analysis group since 1940), Walther first met Zuse in late 1942 [Zuse, 1993].

Wagner, that pioneer of aviation engineering and ingenious manipulator of numbers, had recognised the universal importance of Zuse's computer and had actively supported the project. Zuse wanted to work with Walther on his doctorate on the theme of the theory of general calculation. But Walther at that time regarded the computer primarily as a technical tool for rational engineering science calculations, in the sense of the numerical evaluation of formulas. Zuse's doctorate unfortunately remained only an outline. Petzold suspects that it would have proved difficult to carry out such work with Walther, who gave priority to analogue technology (see [Petzold, 1992, p. 197]).

11.5.4 A structural analysis matrix method: the carry-over method

Nevertheless, Walther, by promoting Zurmühl, had recognised the heuristic power of matrix formulation for physics and the fundamental engineering science disciplines. And therefore the Darmstadt doctorate project of H. Fuhrke on the determination of beam oscillations with the help of matrices could be completed in the early 1950s [Fuhrke, 1955].

Even more important for structural analysis was the carry-over method for calculating continuous beams with any number of spans created by S. Falk in 1956 [Falk, 1956], which translated the solution to the differential beam equation fully into the language of matrix formulation (Fig. 11-38). The carry-over method only exists through matrix operations and in the case of continuous beams leads to systems with a maximum of two linear equations. The degree of static or geometric indeterminacy does not appear in the carry-over method, which belongs to the group of reduction methods; far more significant are the topological properties of the structural system. Consequently, the dual nature of theory of structures – due to the force and displacement methods – is insignificant in the carry-over method.

Joachim Scheer was probably the first engineer in the German-speaking countries to investigate in detail the use of program-controlled automatic calculators for structural tasks in conjunction with the carry-over method [Scheer, 1958]. The program presented by Scheer in 1958 was employed for practical tasks, e. g. a number of projects for the engineering practice of Dr. Homberg in Hagen [Scheer, 1998]. Scheer told the author in 1998 that his dissertation on the problem of the overall stability of singly-symmetric I-beams published in the journal *Der Stahlbau* in 1959 had only been rendered possible through the use of the carry-over method and computers in 1957/1958 [Scheer, 1998]. Despite this, the influence of the carry-over method, like other reduction methods, remained limited in the theory and practice of structural analysis because matrix analysis covered only some of the structural systems. At the same time, Klöppel and Scheer employed matrix analysis successfully for preparing the programming of the buckling theory of stiffened rectangular steel plates ac-

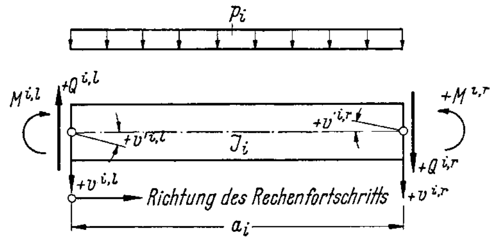


Bild 4. Bezeichnungen und Vorzeichen für Größen am Trägerabschnitt i

Zusammengefaßt kann man in Matrizenform mit

$$\mathfrak{v}^{i,l} = \begin{pmatrix} v^{i,l} \\ v'^{\,i,l} \\ M^{i,l} \\ Q^{i,l} \\ 1 \end{pmatrix} = \text{Vektor am Beginn (= am linken Rand) des Trägerabschnittes } i \quad \ldots\ldots\ldots \text{(4a)}$$

$$\mathfrak{v}^{i,r} = \begin{pmatrix} v^{i,r} \\ v'^{\,i,r} \\ M^{i,r} \\ Q^{i,r} \\ 1 \end{pmatrix} = \text{Vektor am Ende (= am rechten Rand) des Trägerabschnittes } i \quad \ldots\ldots\ldots \text{(4b)}$$

$$\mathfrak{A}^{i} = \begin{vmatrix} 1 & a_i & -\dfrac{a_i^2}{2EJ_i} & -\dfrac{a_i^3}{6EJ_i} & \dfrac{a_i^4 \cdot p_i}{24EJ_i} \\ & 1 & -\dfrac{a_i}{EJ_i} & -\dfrac{a_i^2}{2EJ} & \dfrac{a_i^3 \cdot p_i}{6EJ_i} \\ & & 1 & a_i & -\dfrac{a_i^2 \cdot p_i}{2} \\ & & & 1 & -a_i \cdot p_i \\ & & & & 1 \end{vmatrix} \quad \ldots \text{(4c)}$$

$$\mathfrak{v}^{i,r} = \mathfrak{A}^{i} \cdot \mathfrak{v}^{i,l} \quad \ldots\ldots\ldots \text{(5)}$$

FIGURE 11-38
Carry-over method after Falk in the representation by Scheer [Scheer, 1958, p. 228]

cording to the energy method. With the help of the IBM 704 computer donated to Darmstadt Technical University by IBM Deutschland in 1958, it was possible to calculate the buckling values of standard stiffened rectangular plate cases from the buckling matrix in a relatively short time and publish these as design charts for everyday structural steelwork calculations [Klöppel & Scheer, 1960]; a second volume followed eight years later [Klöppel & Möller, 1968]. Such design charts for the stress analyses of plate and shell structures calculated with the help of sophisticated research programmes provided important assistance in the production of structural calculations carried out partly by hand and partly with the computer even after the innovation phase of theory of structures (1950–1975).

The carry-over method was the historico-logical starting point of structural matrix analysis. This fact is revealed by numerous publications that appeared in the middle of the innovation phase of theory of structures (1950–1975), one example of which was *Matrix Methods in Elastomechanics* (Fig. 11-39). The cover shows a transformation matrix for rotating the system of coordinates through angle γ about the z axis.

The carry-over method was suitable for manual and computerised calculations; this latter point had already been mentioned by S. Falk in 1956 (see [Falk, 1956, p. 231]). The carry-over method could be used to multiply an $m \times r$ matrix (left) by an $r \times n$ matrix (right) in a particularly simple and clear fashion according to the scheme introduced by Falk [Falk, 1951]. The $r \times n$ matrix is positioned to the right above the $m \times r$ matrix such that the extended n columns of the $r \times n$ matrix and the extended m rows of the $m \times r$ matrix overlap to form the result matrix, the $m \times n$ matrix. For example, the element in the ith row and kth column of the result matrix is calculated from the sum of the products of the respective elements in the ith row of the $m \times r$ matrix and the associated elements in the kth column of the $r \times n$ matrix. Fig. 11-40 shows a numerical example of a matrix multiplication according to the Falk scheme. The $m \times r$ matrix ($m = 3$, $r = 2$) is to be multiplied from the right by the $r \times n$ matrix ($r = 2$, $n = 4$). The element in the third line and third column of the $m \times n$ result matrix ($m = 3$, $n = 4$) then becomes $(6 \times 6) + (1 \times 8) = 36 + 8 = 44$. In the Falk scheme the arithmetisation of the matrix calculation for the purpose of programming is obvious; the suitability of the Falk scheme for manual calculations does not contradict this, but ensures that manual calculations, too, undergo further formalisation. Therefore, the prescriptive use of symbols became ever more established in the everyday work of the practising structural engineer.

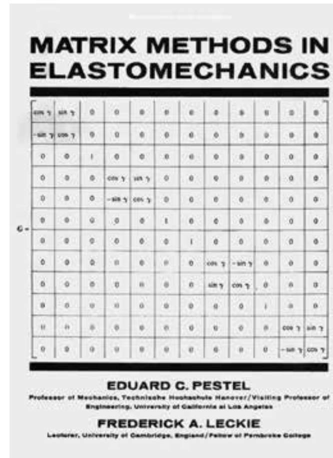


FIGURE 11-39
Cover of the pocket-book edition of *Matrix Methods in Elastomechanics* [Pestel & Leckie, 1963]

The carry-over method is a method for solving linear differential equations of the order $2n$ ($n = 1, 2, 3, 4, \ldots$). The only difference is that the carry-over method is formulated in the language of matrix algebra. Christian Petersen extended the carry-over method significantly. Examples of his work are his derivatives of the transformation matrices for the beam on continuous elastic supports [Petersen, 1965], the curved beam [Petersen, 1966/2] and the circular curved beam on elastic supports [Petersen, 1967]. Nevertheless, the carry-over method is not suitable for solutions with a severely decaying character such as the beam on elastic supports. On the other hand, the carry-over method supplies reliable results when investigating beams with a high bending stiffness. For example, Petersen was the first to specify the right transformation matrices for calculating the eigenfrequencies and eigenmodes of guyed masts modelled as continuous beams on elastic supports [Petersen, 1970]. He established that the shear force Q and the normal force N belonging to the orthogonal section were taken instead of the transverse force T_{iR} and the longitudinal force D_i (from the transverse section), which is totally wrong when formulating the boundary and transfer conditions at the elastic spring supports. Therefore, in his later study on the themes of second-order theory, and also for overturning, torsional-flexural buckling and buckling problems, Petersen derived the basic equations and their solutions always using transverse sections (Fig. 11-41).

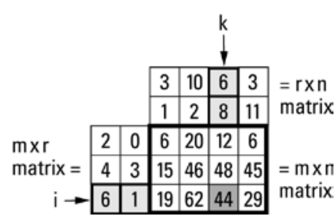
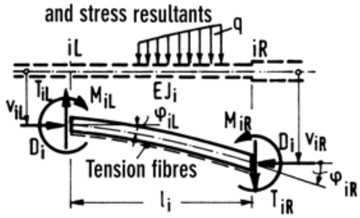


FIGURE 11-40
Numerical example of matrix multiplication according to the Falk scheme

In his habilitation thesis on the vibrations of tower-like structures taking particular account of an attenuation model independent of frequency and stochastic excitation [Petersen, 1971], Petersen determined transformation matrices for a series of problems. This thesis concerns the development of a carry-over method for calculating externally excited

Transformation matrices method, second-order theory

Definition of deformations and stress resultants



Bar factor: $\varepsilon = l\sqrt{\dfrac{D}{EJ}}$ D (compression)

Transformation matrix:

$$[F] = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{bmatrix} \qquad [F]^q = \begin{bmatrix} f_{1q} \\ f_{2q} \\ f_{3q} \\ f_{4q} \end{bmatrix}$$

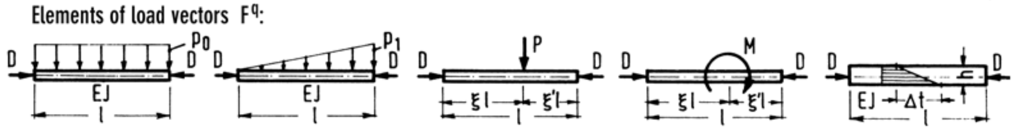
Elements of field matrix F (EJ = const.), first-order theory:

$$[F] = \begin{bmatrix} 1 & l & -\frac{1}{2}\cdot\frac{l^2}{EJ} & -\frac{1}{6}\cdot\frac{l^3}{EJ} \\ 0 & 1 & -\frac{l}{EJ} & -\frac{1}{2}\cdot\frac{l^2}{EJ} \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Second-order theory:

$$[F] = \begin{bmatrix} 1 & \frac{\sin\varepsilon}{\varepsilon}\cdot l & -\frac{1-\cos\varepsilon}{\varepsilon^2}\cdot\frac{l^2}{EJ} & -\frac{\varepsilon-\sin\varepsilon}{\varepsilon^3}\cdot\frac{l^3}{EJ} \\ 0 & \cos\varepsilon & -\frac{\sin\varepsilon}{\varepsilon}\cdot\frac{l}{EJ} & -\frac{1-\cos\varepsilon}{\varepsilon^2}\cdot\frac{l^2}{EJ} \\ 0 & \varepsilon\sin\varepsilon\cdot\frac{EJ}{l} & \cos\varepsilon & \frac{\sin\varepsilon}{\varepsilon}\cdot l \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elements of load vectors F^q:



First-order theory:

$$p_0\cdot\begin{bmatrix} \frac{1}{24}\cdot\frac{l^4}{EJ} \\ \frac{1}{6}\cdot\frac{l^3}{EJ} \\ -\frac{1}{2}\cdot l^2 \\ -l \end{bmatrix} \quad p_1\cdot\begin{bmatrix} \frac{1}{120}\cdot\frac{l^4}{EJ} \\ \frac{1}{24}\cdot\frac{l^3}{EJ} \\ -\frac{1}{6}\cdot l^2 \\ -\frac{1}{2}\cdot l \end{bmatrix} \quad P\cdot\begin{bmatrix} \frac{\xi'^3}{6}\cdot\frac{l^3}{EJ} \\ \frac{\xi'^2}{2}\cdot\frac{l^2}{EJ} \\ -\xi'\cdot l \\ -1 \end{bmatrix} \quad M\cdot\begin{bmatrix} -\frac{\xi'^2}{2}\cdot\frac{l^2}{EJ} \\ -\xi'\cdot\frac{l}{EJ} \\ 1 \\ 0 \end{bmatrix} \quad \alpha_t\frac{\Delta t}{h}\cdot\begin{bmatrix} -\frac{l^2}{2} \\ -l \\ 0 \\ 0 \end{bmatrix}$$

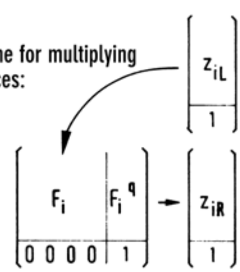
Second-order theory:

$$p_0\cdot\begin{bmatrix} \frac{\varepsilon^2-2(1-\cos\varepsilon)}{2\varepsilon^4}\cdot\frac{l^4}{EJ} \\ \frac{\varepsilon-\sin\varepsilon}{\varepsilon^3}\cdot\frac{l^3}{EJ} \\ -\frac{1-\cos\varepsilon}{\varepsilon^2}\cdot l^2 \\ -l \end{bmatrix} \quad p_1\cdot\begin{bmatrix} \frac{\varepsilon^3-6(\varepsilon-\sin\varepsilon)}{6\varepsilon^5}\cdot\frac{l^4}{EJ} \\ \frac{\varepsilon^2-2(1-\cos\varepsilon)}{2\varepsilon^4}\cdot\frac{l^3}{EJ} \\ -\frac{\varepsilon-\sin\varepsilon}{\varepsilon^3}\cdot l^2 \\ -\frac{l}{2} \end{bmatrix} \quad P\cdot\begin{bmatrix} \frac{\varepsilon\xi'-\sin\varepsilon\xi'}{\varepsilon^3}\cdot\frac{l^3}{EJ} \\ \frac{1-\cos\varepsilon\xi'}{\varepsilon^2}\cdot\frac{l^2}{EJ} \\ -\frac{\sin\varepsilon\xi'}{\varepsilon}\cdot l \\ -1 \end{bmatrix} \quad M\cdot\begin{bmatrix} -\frac{1-\cos\varepsilon\xi'}{\varepsilon^2}\cdot\frac{l^2}{EJ} \\ -\frac{\sin\varepsilon\xi'}{\varepsilon}\cdot\frac{l}{EJ} \\ \cos\varepsilon\xi' \\ 0 \end{bmatrix} \quad \alpha_t\frac{\Delta t}{h}\cdot\begin{bmatrix} -\frac{(1-\cos\varepsilon)}{\varepsilon^2}\cdot l^2 \\ -\frac{\sin\varepsilon}{\varepsilon}\cdot l \\ -EJ(1-\cos\varepsilon) \\ 0 \end{bmatrix}$$

Scheme for transformation from interface iL to interface iR: $[z]_{iR} = [F]_i\,[z]_{iL} + [F]_i^q$. $[z]_i$ is the state vector that includes the deformations and stress resultants at interface i:

$$[z]_i = \begin{bmatrix} v_i \\ \varphi_i \\ M_i \\ T_i \end{bmatrix} \begin{matrix} \text{Deflection} \\ \text{Angle of rotation due to bending} \\ \text{Bending moment} \\ \text{Transverse force} \end{matrix}$$

Scheme for multiplying matrices:



attenuated beam vibrations according to second-order theory, for which he specifies the complex transformation matrix [Petersen, 1971, pp. 95 – 100]. "In the meaning of mathematics," Petersen writes, "the carry-over method achieves exact solutions for various individual problems – something that no FEM calculations achieve. My intention at that time, to write a book about the method of transformation matrices, was abandoned again as the 'heavy-calibre' FEM started to assert itself" [Petersen, 2017, p. 3]. The "heavy-calibre FEM" would first become practically effective as computational statics within the scope of computational mechanics during the diffusion phase of theory of structures (1975 to date).

FIGURE 11-41 (PAGE 844)
Transformation matrices for trusses according to first- and second-order theory on the basis of transverse internal forces T_{iR} and D_i [Petersen, 1980, p. 202]

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