Georgios Gaganelis, Peter Mark, Patrick Forman

Optimization Aided Design

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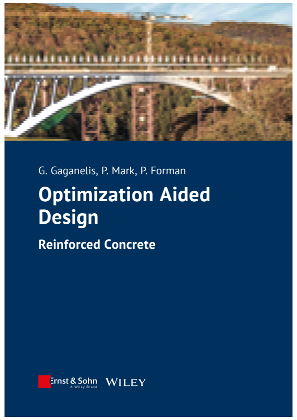

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ABOUT THE BOOK

Optimization Aided Design provides novel methods to use reinforced concrete in a particularly efficient way. Mathematical optimization is applied to the practical problems of concrete design. The aim is to employ the world's most widely used building material in the most economical way and thus substantially reduce CO2 emissions from cement and steel production as well as resource consumption of gravel, sand and water.
Three topics are addressed. First, the identification of the structure. This means the question of the right outer shape such that slender load-bearing designs develop following the flux of forces. In line with the stress affinity of the material, the structures are predominantly subjected to compression. Second, the reinforcement layout, which is oriented to the stress trajectories. Advantages arise particularly for walls, voluminous structural components, load introduction areas and cut-outs. Clear strut-and-tie models emerge that are directly convertible into reinforcement layouts. Third, the treatment of cross-sections. They are optimized in their shape and designed in their reinforcement. This also applies to sophisticated loading conditions (biaxial bending) and virtually arbitrary geometrical configurations. Parameterization allows the transfer to general cross-section types.
The optimization aided methods are described extensively and in an illustrative manner. They are universally applicable and independent of standards, concrete types and reinforcements. They apply to normal strength to ultra-high performance concretes, to reinforcements made of steel, carbon or glass fibers, and to rebars as well as reinforcing fibers. Numerous illustrations and computation examples demonstrate their application. Moreover, practical applications are presented, including ultra-light concrete-steel beams, slender concrete solar collectors, and improved reinforcement layouts for tunnel lining. The book addresses students, researchers, and practitioners alike.

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Foreword by Manfred Curbach

Intensifying Creativity in Construction

There is hardly a topic among building professionals that is discussed more intensively than sustainable construction. In view of the emphasis on this topic, it appears that intensive work is being done on the implementation of this challenge, both in research and in realization. After all, it is about nothing less than building in a way that enables all people of the generations to come to live a decent life on this earth. Because we have only this one. In 1994, the astronomer and astrophysicist Carl Sagan had the idea of taking a photo of the Earth with the help of the Voyager 1 space probe after it left the solar system. In a lecture on 13 October 1994 at the Cornell University, he said the following about this:

> Our planet is a lonely speck in the great enveloping cosmic dark. In our obscurity, in all this vastness, there is no hint that help will come from elsewhere to save us from ourselves. There is perhaps no better demonstration of the folly human conceits than this distant image of our tiny world. To me, it underscores our responsibility to deal more kindly with one another, and to preserve and cherish the pale blue dot, the only home we've ever known.[1]

In fact, we are overexploiting and consuming the resources of our earth and changing them massively. The consequences are climate change, scarcity of resources, natural disasters, hunger, flight, and misery. And the construction industry is massively involved in these developments.

The building industry in Germany accounted for 5.3% of nominal gross value added in 2018 (€179.6 billion GDP of €3388.2 billion GDP)[2] but causes around 25% of CO_2 emissions and uses around 40% of the energy generated.[3]

This discrepancy alone should lead to enormous productive activities. But what is the reality in terms of efficiency and research?

1 Sagan, C. (1997). *Pale Blue Dot*. United States: Random House USA Inc.
2 Statistisches Bundesamt (2019). *Bruttoinlandsprodukt 2018 für Deutschland*. Wiesbaden: Statistisches Bundesamt.
3 Hong, J., Shen, G.Q., Feng, Y., et al. (2015). Greenhouse gas emissions during the construction phase of a building: a case study in China. *Journal of Cleaner Production* 103: 249–259.

In sectors such as manufacturing (excluding construction), productivity increased by around 70% from 1995 to 2016, whereas in construction it only increased by around 5%.[4]

In terms of industry investment in research and development, out of a total of 436 571 people (in full-time equivalents) in 2017, only 1147 people came from the construction industry, i.e. 0.26%.[5]

The Federal Government of the Federal Republic of Germany spent a total of €17 250 million on research and development in 2018. Of this, €118.1 million was allocated to the area of "Regional planning and urban development; construction research," i.e. 0.69%.[6]

Considering only the Federal Ministry of Education and Research, a total of €10 486.7 million was invested in 2018. The area of "Regional planning and urban development; construction research" accounted for a share of only €27.5 million, i.e. 0.26%.

In 2019, the annual grant total from the German Research Foundation amounted to around €3285.3 million. The field of Civil Engineering and Architecture received grants totaling €51.5 million, i.e. 1.57%.[7]

The result of this small survey illustrates that in one of the most important industries in Germany, which contributes disproportionately to climate change, efficiency is stagnating and, at the same time, research is receiving severely below-average funding.

Every 12 years, the population of the earth grows by 1 billion people[8] who need a decent home, infrastructure, and energy supply. In view of the continuing increase in the world's population, we will not build less, but more. Contrary to this, we need to radically limit resource consumption and CO_2 emissions. It is obvious that in the future, building will have to be completely different, not just marginally, but fundamentally.

It is thus clear that we must significantly intensify research in the construction industry. Because of its enormous leverage effect, this is therefore one of the most important tasks for the future, both nationally and internationally, with extremely great significance for society as a whole. At all levels, from basic research to realization, for all available and newly to be developed building materials and combinations of building materials, in all areas of our social life up to politics, we have to become much more creative. Only through our inventiveness, our power of imagination for realization, our abilities to mentally penetrate complex processes

---

4 Statistisches Bundesamt (2020). Inlandsproduktberechnung – Lange Reihen ab 1970, Fachserie 18 Reihe 1.5. https://www.destatis.de/DE/Themen/Wirtschaft/Volkswirtschaftliche-Gesamtrechnungen-Inlandsprodukt/Publikationen/Downloads-Inlandsprodukt/inlandsprodukt-lange-reihen-pdf-2180150.html (accessed 27 August 2021).

5 Stifterverband für die Deutsche Wissenschaft e.V. (2019). *Forschung und Entwicklung in der Wirtschaft 2017*. Essen: SV Wissenschaftsstatistik GmbH.

6 Bundesministerium für Bildung und Forschung (2020) Bildung und Forschung in Zahlen 2020 (It should be noted that the figures therein for 2019 and 2020 are target figures).

7 Köster, T.; Lüers, K.; Schneeweiß, U.; Hohlfeld, C.; Kaufmann-Mainz, N. (2020). Deutsche Forschungsgemeinschaft - Jahresbericht 2019 – Aufgaben und Ergebnisse.

8 United Nations (2019). World Population Prospects: The 2019 Revision. https://www.dsw.org/infografiken/ (03 April 2021).

will we change the entire building process from design, planning, calculation, structure, material extraction, production, transport, on-site construction, operation, maintenance, data storage, strengthening up to further use, reuse, and recycling in such a way that we achieve climate- and resource-neutral building.

The methods, procedures, and calculations described in this book represent an important step toward a kind of building that has little to do with the way we know it today. And this is a good thing.

At the same time, may this book also promote the idea that it is worthwhile for everyone to think about change in the building industry to contribute ideas, to conduct research, and to work on realization. May the amount of research increase to a degree that is both appropriate and necessary to the challenge we all face.

Dresden, June 2021 *Manfred Curbach*

Prof. Dr.-Ing. Dr.-Ing. E.h. Manfred Curbach. Since 1994 professor for concrete structures at the Technische Universität Dresden, from 1999 to 2011 speaker of the SFB 528 "*Textile Reinforcement for Structural Strengthening and Repair,*" *since 2013 speaker of the BMBF consortium C^3 – Carbon Concrete Composite, since 2020 speaker of the SFB/TRR 280* "*Design Strategies for Material-Minimised Carbon Reinforced Concrete Structures,*" *2014 foundation of the company CarboCon for the practical implementation of carbon concrete, 2016 winner of the German Future Prize, Award of the Federal President for Technology and Innovation.*

Foreword by Werner Sobek

Building Emission-Free for More People with Less Material

Concrete is the only building material that can be cast into almost any shape on the construction site. It is available worldwide, is cheap, is easy to use, has a comparatively high strength, and is resistant to most environmental conditions. Concrete is the building material for everyone and for everything; it is the most widely used building material in the world. On the other hand, concrete is more unpopular for most people than almost any other material. In the past decades, this dislike was mainly based on its color, the quality of its surface, and its "coldness" (i.e. its low heat radiation). Today, it is the massive criticism of the CO_2 emissions caused by the production of cement, which, at around 8% of global CO_2 emissions, make a significant contribution to global warming and which, in terms of volume, even exceed the emissions of the entire global air traffic that labels concrete as an unloved, even demonized building material.

The currently widespread message that it will be possible to replace concrete as a building material with timber in the short to medium term is mostly based on an ignorance of the interrelationships. Mankind currently needs approximately 60–100 Gt of building materials per year in order to create a home for the new inhabitants on earth and to expand the existing built environment. This number does not include the so-called pent-up demand of the Third World, which with 6.3 billion people represents approx. 80% of the world's population and which, with a volume of approx. 60 t per capita, has a significantly lower building standard than the citizens of the industrialized nations, who account for approx. 335 t of building materials per capita.

If all the forests in the world were managed according to the principles of sustainable forestry, as is the case in many countries in Central Europe, for example, then a maximum of 10 Gt of construction timber could be obtained per year. An increased supply of construction timber through a higher logging rate in the forests would mean a reduction of the urgently needed CO_2 sink potential of these forests and must therefore be rejected. A redistribution of the available timber toward the construction industry would mean a reduction in the availability of wood to produce cellulose, paper or, for example, the abandonment of the cooking of daily food, as is still the case for many hundreds of millions of people every day.

To the scenario described above, the already mentioned pent-up demand for the inhabitants of the Third World. If the frequently voiced demand for an increase in prosperity and thus also a reduction in the birth rate in these countries were actually to be met through easier access to health care and education, especially for the female part of the population, the corresponding construction activities would have to be carried out, for example the building of schools and universities, medical practices and hospitals, including the associated infrastructure. Raising the level of construction in the Third World countries to that of today's industrialized countries can be estimated with a demand for building materials of 1700 Gt. This amount of building material represents twice the world built today. The climate-damaging emissions associated with the production of these building materials would make the earth uninhabitable for mankind. It is therefore evident that we will not be able to raise the total population of this earth to the building level of today's industrialized countries nor will timber as a building material be able to play a significant role in this context in the short to medium term. Timber will be an important building component in some parts of the world, especially in the Northern Hemisphere, in the short to medium term. Not more. Other building materials, such as clay or natural stone, will also increasingly find their way into construction. However, none of these materials will be able to replace concrete as a building material.

But what should the builders, the architects, the engineers, and the executing companies do, on the one hand, to fulfill their responsibility to provide a built home, including all the necessary infrastructural construction measures, for more and more people and, on the other hand, to make their enormously important contribution to limiting, even reducing, global warming? Since an ideal way has not yet been identified and the marvel material that solves all problems has not yet been found, the solution to the overall problem will consist of a sum of components. One of these components is the restriction of construction activities to what is actually necessary, the appropriate amount. This is often referred to as the principle of sufficiency. The principle of sufficiency includes the requirement not to demolish buildings or parts of them and replace them with new buildings until this is really unavoidable.

Another component of sustainable construction is the revolutionization of construction technology to the effect that in the future only recycling-oriented planning and construction will be permitted. In this way, the extraction of new building materials from the upper layers of the earth can be increasingly reduced in the medium to long term. This will also diminish, if not solve, the availability problems of individual building materials. It is common knowledge that enormous quantities of sand and gravel are required, especially for the production of concrete, and that sand has already become an extremely rare resource in some regions of our planet. The same applies to gravel and crushed stone. Immense availability problems are also expected for tin, zinc, and copper. For the construction industry, being the largest consumer of resources of all, it is therefore a matter of dramatically reducing the "consumption" of primary materials in the future and of using secondary materials where they are actually unavoidably needed. If we consider the availability of resources in addition to the emissions, it can be seen that the local and regional production of secondary material is associated with significantly lower climate-damaging emissions compared to primary material that is often delivered over long transport routes.

While the implementation of the closed-loop principle reduces the amount of "consumed" primary building materials, the complementary implementation of lightweight construction technologies can reduce the amount of consumed material and the amount of climate-damaging emissions during its production and distribution. This is where this book comes in. The introduction of state-of-the-art optimization methods and the resulting minimum-material component shapes, which also have a minimized need for reinforcing steel due to optimized reinforcement design, promote construction with concrete that is characterized by considerable material savings and thus considerable emission savings for the same utility value and durability. Supported by clearly understandable descriptions and a large number of examples, readers will find their way around quickly and easily. This makes it much easier to understand the subject matter, which is not always simple.

This book provides a significant contribution to establishing a new foundation for building with concrete, this wonderful building material for everyone and for almost everything. This foundation is characterized by the application of highly developed calculation methods and technologies that lead to material-minimized components and thus also to emission-minimized components. Both will be an essential part of tomorrow's construction, a construction that, like other sectors such as transportation or energy, must reduce its emissions by more than 50% by 2030. No one knows today how this will ultimately be achieved. However, the paths outlined in this book represent a valuable and indispensable tool on the way to achieving these goals.

Stuttgart, June 2021 *Werner Sobek*

Prof. em. Dr. Dr. E.h. Dr. h.c. Werner Sobek. From 1995 to 2021 successor of Frei Otto, and at the same time, from 2000 to 2021 also successor of Jörg Schlaich at the University of Stuttgart. Founder of the ILEK Institute of lightweight structures and conceptual design at the University of Stuttgart. From 2008 to 2014 professor at the Illinois Institute of Technology in Chicago in the succession of Ludwig Mies van der Rohe. Founder of the Werner Sobek group. Founder and co-founder of several charitable foundations and non-profit associations in the field of construction.

Preface

This book is based on over 15 years of research work at the Institute of Concrete Structures (Ruhr University Bochum, Germany) on topics related to structural optimization and lightweight concrete structures. The motivation, then and now, derives from two fundamental reasons. First, the climate challenge and the related necessity for lower material consumption. Second, modernizing the construction industry through new technologies aiming at more sustainable design and construction methods. The concepts evolved from the research work are combined in this book into an enhanced design approach, which we call *Optimization Aided Design* (OAD).

From students to researchers and practitioners, this book addresses everyone involved in structural engineering. Although the concepts primarily focus on concrete structures, they are generally adaptable to a wide range of further applications regardless of the material used. Numerous computational examples serve for a better comprehension of the methods and invite to discover the potential of OAD. Applications that have been successfully implemented further demonstrate transferability in practice and intend to provide inspiration for future projects.

Apart from the introduction, the book consists of two parts. The first part serves as introduction to the fundamentals of reinforced concrete design, on the one hand, and structural optimization on the other. Chapters 2 and 3 provide a general basis for understanding the methods presented subsequently. In no case do they claim to be exhaustive. For a more in-depth study of both topics, many excellent books from other colleagues already exist. In this regard, reference is made to the bibliography. The second part of the book introduces OAD for concrete structures. The methods are presented structurally from the outside in. In doing so, first, approaches for identifying the external structural shape are presented, followed by methods for designing the inner one (reinforcement layout), and finally techniques for the optimization of cross-sections.

Each of the OAD chapters is divided into three parts. They begin with a brief topical description supported by a representative overview figure, allowing the reader to decide whether the subsequent content has relevance for her or him. This is followed by the main section, in which the methods are discussed exhaustively and are supplemented with recommendations for their practical application. Numerous computation examples, to which reference is made in the respective main sections, provide the conclusion. They are further enhanced by application examples which have already been realized, for example ultra-light beams, extremely thin shells of solar thermal power plants or optimized reinforcement layouts for segmental tunnel linings.

OAD offers the possibility to enhance the daily engineering work and increase its efficiency. Our ambition is to highlight the great potential of the approach and thereby contribute to a modern, sustainable, and transparent way of designing and dimensioning reinforced concrete structures in the future. However, this can only succeed if we open the door for modern approaches and thus prove to the new generation, that the construction industry is able to adapt to the modern age. Considering the global challenges, let us be part of the solution, not part of the problem.

Bochum, Germany
April 2021

Georgios Gaganelis
Peter Mark
Patrick Forman

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4

Identification of Structures

Key learnings after reading this chapter:

- What are practical approaches for identifying global optimal structure designs?
- What are typical results and how do different optimization parameters influence them?
- How can optimization results be transferred into structure designs?

This chapter discusses optimization methods as a tool for two types of problems (Figure 4.1).

The first is the identification of a good structure design for a given building material within a permissible design space and under consideration of static and geometrical boundary conditions in the draft phase. The principles are threefold: resource efficiency, cost-effectiveness, and material compatibility. For the latter, it is particularly important to ensure that the designs are adapted to the material-specific properties, namely compression or tension affinity.

The second type of problem is to improve the overall design of standard components such as beams, girders, and walls. Starting from the given basic shape, the aim here is to reduce material consumption without compromising stiffness and load-bearing capacity. For this purpose, the amount of material available to form the structure is first reduced and then rearranged toward the internal load transfer (principal stress trajectories). The results yield lightweight reinforced concrete (RC) and concrete–steel hybrid structure designs.

Optimization Aided Design: Reinforced Concrete, First Edition.
Georgios Gaganelis, Peter Mark, and Patrick Forman.

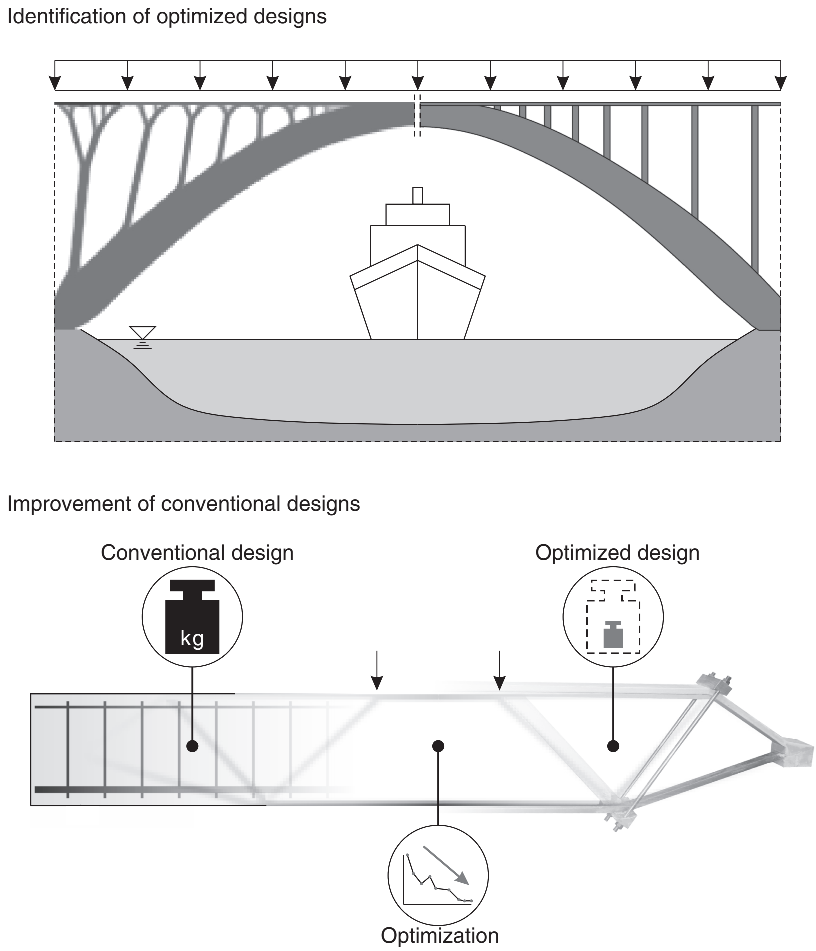

Figure 4.1 Overview of Chapter 4: identification of structures.

4.1 One-material Structures

Related Examples 4.1–4.10.

4.1.1 Problem Statement

In topology optimization, a limited amount of material is iteratively redistributed within a predefined design space in such a way that the objective function is minimized until a convergence criterion is met. The material distribution is oriented to the internal force flow according to the “form follows force” principle. Using continuum finite elements (continuum topology optimization, CTO), topology optimization can be applied to reduce a structure’s material consumption to the necessary load-bearing system by eliminating any excess material. In this way, resource efficiency is aimed at already at the (pre-)design stage. This is particularly important for RC structures, since their main components, namely cement, concrete, and reinforcing steel, emit vast quantities of greenhouse gas (GHG) in production. For classification, the construction industry accounts for 25 % of worldwide CO_2 emissions [1], with cement alone accounting for 5–10 % [2–5] as already mentioned in Section 1.3. In this regard, CTO can be used as a design tool for distinctly more sustainable structures [6–8].

Typically, the optimization problem's objective function is defined to minimize the mean structural compliance (c), which equals stiffness maximization. In discrete finite element (FE) notation, the compliance reads

$$c = \mathbf{FU} = \mathbf{U}^\top \mathbf{KU} \tag{4.1}$$

where $\mathbf{F}$, $\mathbf{U}$, and $\mathbf{K}$ are the global load vector, displacement vector, and stiffness matrix, respectively. The available material volume (V), which is limited to a fraction $\beta \in [0, 1]$ of the initial volume (V^0), serves as constraint in order to prevent trivial solutions in which the design space is completely filled with material to achieve maximum stiffness.

The FE analysis model is linked to the optimization problem via material interpolation schemes [9]. The most popular is SIMP (Solid Isotropic Material with Penalization) [9–14]. In SIMP, each element e is assigned a design variable $x_e \in [0, 1]$, where the latter can be considered normalized element densities:

$$x_e = \frac{\rho_e}{\rho_e^0} \tag{4.2}$$

In Eq. (4.2), ρ_e is the associated element density and ρ_e^0 is the physical density of the employed material. Through stiffness adjustment of each element, the design variables define also their material assignment:

$$E_e(x_e) = x_e^p E^0 \tag{4.3}$$

where E_e and E^0 are the elemental and physical Young's modulus, respectively. The penalty exponent $p > 1$ gives preference to a 0-1 distribution of the design variables by assigning underproportional stiffness to intermediate values of the design variables (Figure 4.2). In this way, they become inefficient for minimizing the structural compliance, hence they are avoided by the optimization algorithm. This is reasonable, because intermediate densities are difficult to interpret in practice [15]. A reasonable value for p can be justified physically with respect to the Poisson's ratio of the employed material [9, 11]. However, generally it suffices to simply set $p = 3$.

The optimization problem can then be expressed comprehensively as follows:

$$\begin{aligned}
\text{find:}\quad & \boldsymbol{x} = [x_1, x_2, \ldots, x_{N_e}]^\top \\
\text{such that:}\quad & f(\mathbf{x}) = c = \mathbf{U}^\top \mathbf{K}(\mathbf{x})\mathbf{U} = \textstyle\sum_{e=1}^{N_e} \mathbf{u}_e^\top \mathbf{k}_e(x_e)\mathbf{u}_e \to \min_{\mathbf{x}} \\
\text{subject to:}\quad & g(\mathbf{x}) = V - \beta V^0 = \textstyle\sum_{e=1}^{N_e} x_e v_e - \beta \sum_{e=1}^{N_e} v_e \leq 0 \\
& 0 < x_e^L \leq x_e \leq x_e^U \qquad\qquad e \in [1, N_e]
\end{aligned} \tag{4.4}$$

In Eq. (4.4), N_e is the number of elements, $\mathbf{k}_e$ and $\mathbf{u}_e$ are the element's stiffness matrix and displacement vector, respectively, v_e is the volume of element e, $x_e^L = 10^{-3}$ is a

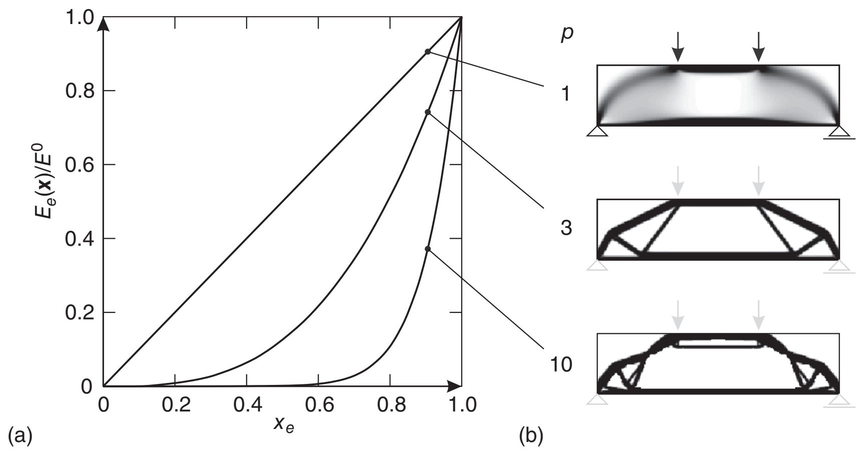

Figure 4.2 Penalty exponent p in SIMP: (a) stiffness assignment, (b) impact on results.

lower nonzero bound of the design variables to avoid singularity and $x_e^U = 1$ is the upper bound, which represents the physical density.

4.1.2 Sensitivity Analysis

The partial derivatives of both the objective and the constraint function, the so-called sensitivities, are required to solve the optimization problem and update the design variables. They can be derived following the adjoint method. For further details, the reader is referred to, for instance, Bendsøe and Sigmund [11]. In doing so, the adjoint method approach leads to:

$$\frac{\partial f(\mathbf{x})}{\partial x_e} = -\mathbf{U}^\top \frac{\partial \mathbf{K}(\mathbf{x})}{\partial x_e} \mathbf{U} = -p x_e^{(p-1)} \mathbf{u}_e^\top \mathbf{k}_e \mathbf{u}_e \tag{4.5}$$

for the objective function sensitivities. The sensitivities of the constraint function, on the other hand, are computed straightforwardly to simply:

$$\frac{\partial g(\mathbf{x})}{\partial x_e} = v_e \tag{4.6}$$

4.1.3 Filtering

Two well-known numerical problems are encountered in CTO [11, 15, 16]. The first is the so-called checkerboard problem. These checkerboard patterns refer to elements containing material, which are connected at their vertices (hinged elements) and cause an oscillating material distribution as shown in Figure 4.3a. The reason for this lies in a numerically overestimated stiffness of such hinged elements when using linear shape functions for approximation in the FE model.

The second problem is known as mesh dependency of the optimization results. If the number of elements forming the FE mesh increases, the resultant structures show more delicate struts and smaller holes as they exhibit higher stiffness values

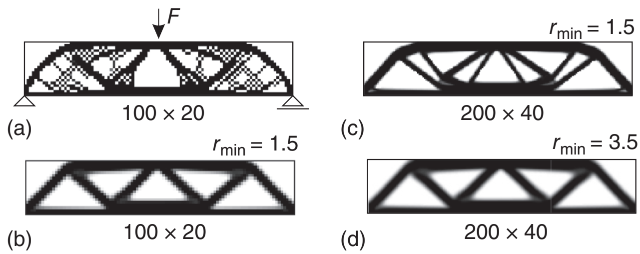

Figure 4.3 Numerical problems in topology optimization: (a) checkerboards, (b, c) mesh dependency, (d) mesh independent filtering.

and are therefore preferred by the optimization algorithm. Thus, instead of a finer FE mesh providing a more accurate numerical approximation of the mechanical behavior, it instead affects the optimization result which is highly undesired. This effect is demonstrated in Figure 4.3b–c where the optimization results for a design space of 100 × 20 and 200 × 40 elements, respectively, are compared.

Various techniques have been developed to overcome these problems [11, 15, 17]. The most popular are the so-called mesh independent filters, inspired by image processing [16]. From a mathematical point of view, by using filters within the optimization, the permissible solution space is limited to a subspace through the applied filter radius $r_{\min}$. From an engineering point of view, however, $r_{\min}$ defines a lower limit for the permissible thickness of structural components forming the resulting structure, cf. Figure 4.3d. The filter radius can therefore be interpreted as a kind of a manufacturing constraint.

Among a multitude of others, the most important filters are the so-called density and the sensitivity filter [11, 15]. For the sake of simplicity, the following explanations are limited to the sensitivity filter [18] since it is easy to implement and sufficient for practical application. It modifies the sensitivities of the objective function at the end of an iteration, right before the design variable update as can be seen from the flow chart in Figure 4.4. The modified sensitivities of an element are then calculated as the mean value of all adjacent elements' sensitivities, lying within $r_{\min}$ and weighted by their distance. Thus, the modified objective function sensitivity of an element e is

$$\frac{\partial \tilde{f}}{\partial x_e} = \frac{1}{x_e \sum_{i=1}^{N_e} H_{ei}} \sum_{i=1}^{N_e} H_{ei} x_i \frac{\partial f}{\partial x_i} \tag{4.7}$$

where

$$H_{ei} = \begin{cases} r_{\min} - \Delta_{ei} & \text{if } \Delta_{ei} \leq r_{\min} \\ 0 & \text{else} \end{cases} \tag{4.8}$$

with Δ_{ei} being the distance between the center of element i and e.

The sensitivity filter is easy to implement since the optimization problem remains unchanged. It should be emphasized that it works heuristically and is not derived mathematically well founded. For practical applications, however, the advantage of simplicity prevails.

4.1.4 Solving

The optimization problem can be solved using a nonlinear optimization algorithm or, alternatively, by applying a heuristic update scheme for the design variables, which is developed using an optimality criterion (OC) derived from the Lagrangian function. The latter is a customized approach to the problem, which has the advantage of being numerically robust and converging to a solution rapidly [6, 19]. For the sake of efficient practical application, the OC-based update scheme is described in more detail below.

The Lagrangian function of the optimization problem stated in Eq. (4.4) reads

$$L = \mathbf{U}^\top \mathbf{K}\mathbf{U} + \Lambda \left(V - \beta V^0\right) + \sum_{e=1}^{N_e} \lambda_e \left(x_e^L - x_e\right) + \sum_{e=1}^{N_e} \gamma_e \left(x_e - x_e^U\right) \tag{4.9}$$

where $\Lambda \geq 0$, $\lambda \geq 0$ and $\gamma_e \geq 0$ are the Lagrangian multipliers. From the stationarity condition with respect to the design variables follows:

$$\frac{\partial L}{\partial x_e} = \left(\frac{\partial \mathbf{U}^\top}{\partial x_e}\mathbf{K}\mathbf{U} + \mathbf{U}^\top \frac{\partial \mathbf{K}}{\partial x_e} + \mathbf{U}^\top \mathbf{K}\frac{\partial \mathbf{U}}{\partial x_e} \right) + \Lambda \frac{\partial g}{\partial x_e} - \lambda_e + \gamma_e = 0 \tag{4.10}$$

Assuming that the loads are independent of the design variables, it can be shown [11] that

$$\mathbf{U}^\top \mathbf{K}\frac{\partial \mathbf{U}}{\partial x_e} = -\mathbf{U}^\top \frac{\partial \mathbf{K}}{\partial x_e}\mathbf{U} \tag{4.11}$$

as well as

$$\mathbf{U}^\top \mathbf{K}\frac{\partial \mathbf{U}}{\partial x_e} = \frac{\partial \mathbf{U}^\top}{\partial x_e}\mathbf{K}\mathbf{U} \tag{4.12}$$

holds true. Substituting Eqs. (4.11) and (4.12) in (4.10) leads to the reformulated stationarity condition:

$$\frac{\partial L}{\partial x_e} = \underbrace{-\mathbf{U}^\top \frac{\partial \mathbf{K}}{\partial x_e}\mathbf{U}}_{\frac{\partial f}{\partial x_e}} + \Lambda \frac{\partial g}{\partial x_e} - \lambda_e + \gamma_e = 0 \tag{4.13}$$

The OC is then obtained by rearranging Eq. (4.13) which finally yields

$$\underbrace{\frac{-\frac{\partial f}{\partial x_e}}{\Lambda \frac{\partial g}{\partial x_e}}}_{G_e} = 1 - \frac{\lambda_e}{\Lambda v_e} + \frac{\gamma_e}{\Lambda v_e} \tag{4.14}$$

By evaluating Eq. (4.34), an update scheme for the design variables can be derived as described in [11, 12, 20, 21], such that G_e approaches 1. Such an update scheme might read, for instance, as follows:

$$x_e^{(k+1)} = \begin{cases} M_x^- & \text{if} \quad \left[G_e^{(k)}\right]^{0.5} x_e^{(k)} \leq M_x^- \\ M_x^+ & \text{if} \quad \left[G_e^{(k)}\right]^{0.5} x_e^{(k)} \geq M_x^+ \\ \left[G_e^{(k)}\right]^{0.5} x_e^{(k)} & \text{else} \end{cases} \tag{4.15}$$

where

$$M_x^- = \max\left\{\left(1-\mu_x\right)x_e^{(k)}, x_e^L\right\} \tag{4.16a}$$

$$M_x^+ = \min\left\{x_e^U, \left(1+\mu_x\right)x_e^{(k)}\right\} \tag{4.16b}$$

Here, x_e^L and x_e^U are the lower and upper bound of the design variables, μ_x is a move limit, which prevents too large changes of the design variables between iterations to avoid convergence problems and k denotes the current iteration number. Appropriate values for the parameters are, for example, $x_e^L = 10^{-3}$, $x_e^U = 1$, and $\mu_x = 0.2$. The Lagrange multiplier Λ in Eq. (4.34) is computed numerically using a bisection algorithm in such a way that the volume constraint is met, since it can be expected that the stiffest structure is obtained by exploiting the maximum amount of permissible material.

4.1.5 Optimization Process

Figure 4.4 shows the flow chart of the one-material topology optimization approach. First, the model is initialized, i.e. the design space, boundary conditions, material properties, and optimization parameters are defined. Then, the first FE analysis is conducted and the corresponding displacement field is computed. From this, the principal stresses and strains are then determined. The associated sensitivities of the objective and constraint functions are then evaluated. In order to counteract well-known numerical problems in optimization with linear elastic FE models, the former are filtered. The sensitivity information of both objective and constraint

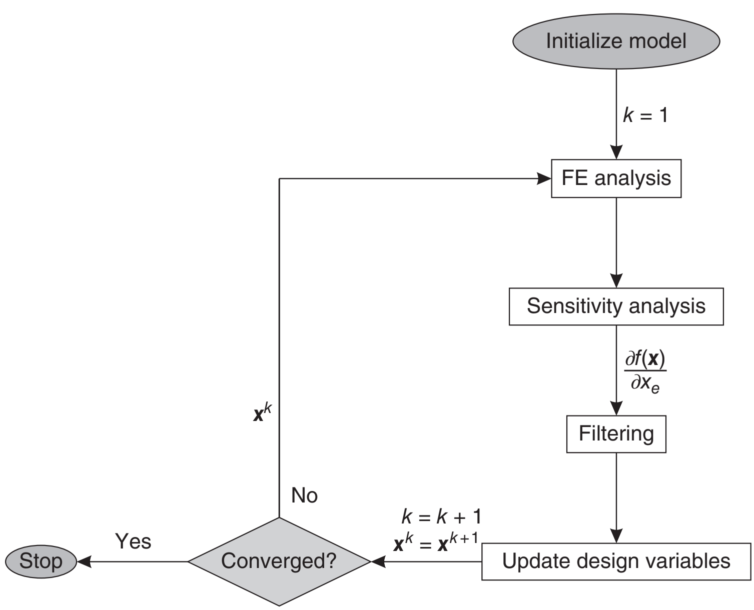

Figure 4.4 Flow chart of the topology optimization approach.

function are then used by the optimizer, for instance, the introduced OC-based update scheme, to update the design variables. Finally, at the end of the iteration, a convergence criterion is checked. A suitable one could, for example, be the largest difference between the design variables of two subsequent iterations:

$$\max \, \Delta x_e = \max \left| x_e^{(k+1)} - x_e^{(k)} \right| \leq \mathrm{tol}_x \tag{4.17}$$

with $\mathrm{tol}_x = 10^{-2}$ being a reasonable limit. If the convergence criterion is met, the optimization is terminated. If not, a new loop is initiated by performing a linear elastic FE analysis.

4.1.6 Multiple Load Cases

Usually, it is necessary to consider more than one load case when determining internal forces and designing structural components. The same applies to the optimization of the structural design, where the existing load cases must also be adequately accounted for. The optimized material distribution differs significantly depending on whether all loads are considered within one or within individual load cases. The single load case results are often unstable designs consisting of rectangular substructures, whereas a multiple load case approach yields stable structures composed of triangular segments, cf. Figure 4.5. The two forces F work like dead loads and act at the same time (a). On the contrary, F_1 and F_2 may occur individually from each other, so both loads or just one of them might be present in a sense of live loads (b).

In the case of the standard compliance minimization approach (Eq. (4.4)), multiple load cases can be taken into account by reformulating the objective function as minimizing the weighted sum of the compliance values obtained from all individual load cases [11]:

$$f(\mathbf{x}) = \sum_{l=1}^{M} w_l c_l = \sum_{l=1}^{M} w_l \mathbf{U}_l^\top \mathbf{K}(\mathbf{x}) \mathbf{U}_l \rightarrow \min_x \tag{4.18}$$

Here, l represents the load case number, M is the total number of all load cases, w_l are the corresponding weighting factors, c_l are the compliance values, $\mathbf{U}_l$ are the respective displacement vectors, and $\mathbf{K}(\mathbf{x})$ is the common global stiffness matrix. The displacement vectors are computed independently from each other, from which then the compliance of each corresponding load case is determined. In turn,

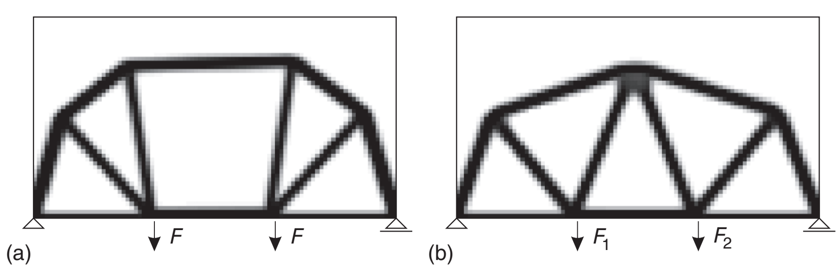

Figure 4.5 Different optimization results depending on the load case definition: (a) one load case, (b) two load cases.

the compliance values are then multiplied by the respective weighting factor, if necessary, and summed up to the total objective function value. Its sensitivities are determined in the same way as weighted sum:

$$\frac{\partial f(\mathbf{x})}{\partial x_e} = \sum_{l=1}^{M} w_l \left(-\mathbf{U}_l^\top \frac{\partial \mathbf{K}(\mathbf{x})}{\partial x_e} \mathbf{U}_l \right) \tag{4.19}$$

The multiple load case problem can be solved with common nonlinear optimization algorithms as well as with the OC-based update scheme given in Eq. (4.15) by substituting Eqs. (4.18) and (4.19) into the objective function and its sensitivities.

4.2 One-material Stress-biased Structures

Related Example: 4.11.

4.2.1 Problem Statement

The main building materials, concrete and steel, show different mechanical properties. Concrete, on the one hand, exhibits a pronounced compressive strength but negligible tensile load-bearing capacities. Due to its very low price and worldwide availability, however, it is the dominant building material and is reinforced by steel reinforcement in structural areas subject to tensile stresses. Steel, on the other hand, has good load-bearing properties in both compression and tension. But because of its high costs, its use is rather reasonable for components predominantly exposed to tensile loading. The global structural design should therefore be material-dependent, i.e. compression-dominant when using concrete and tension-dominant when using steel. Therefore, topology optimization approaches are needed that produce compression- or tension-dominant structural design proposals.

The optimization problem is set up as classical compliance minimization with volume constraint using continuum elements:

$$\begin{aligned}
\text{find:} \quad & \boldsymbol{x} = [x_1, x_2, \dots, x_{N_e}]^\top \\
\text{such that:} \quad & f(\mathbf{x}) = c = \mathbf{U}^\top \mathbf{K}(\mathbf{x}) \mathbf{U} = \textstyle\sum_{e=1}^{N_e} \mathbf{u}_e^\top \mathbf{k}_e(x_e) \mathbf{u}_e \to \min_{\mathbf{x}} \\
\text{subject to:} \quad & g(\mathbf{x}) = V - \beta V^0 = \textstyle\sum_{e=1}^{N_e} x_e v_e - \beta \sum_{e=1}^{N_e} v_e \leq 0 \\
& 0 \leq x_e \leq 1 \qquad\qquad e \in [1, N_e]
\end{aligned} \tag{4.20}$$

Here, $\mathbf{x}$ is the vector containing all N_e design variables, each representing a relative element density, where $x_e = 0$ indicates a void and $x_e = 1$ a solid element with full material allocation, c is the mean structural compliance (reciprocal of structural stiffness), $\mathbf{U}$ and $\mathbf{u}_e$ are the global and local displacement vectors, respectively, V is the structural volume and V^0 is the design space's initial volume, where $\beta \in [0,1]$ represents their ratio and defines the targeted residual volume for the optimized structure, v_e is the volume of element e, $\mathbf{K}$ and $\mathbf{k}_e$ are the global and local stiffness